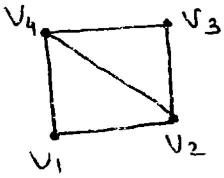
Graph Theory

Graph :- A graph G is defined by $G = (V, E)$ where



V = set of all vertices in G

E = set of all edges in G

$|V|$ = no. of vertices in G = order of G

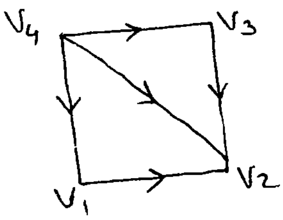
$|E|$ = no. of edges in G = size of Graph

Nondirected (Undirected) graph

- In a nondirected graph, an edge is represented by set of two vertex

$\{V_i, V_j\}$ = An edge betn V_i & V_j

Directed Graph :- (Diagraph)

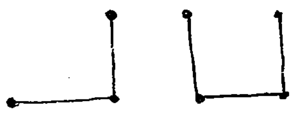


- In directed graph, an edge is represented by an ordered pair of two vertex.

(V_i, V_j) = An edge from V_i to V_j

Connected Graph :-

- A graph is said to be connected if there exist a path betn every pair of vertices.
- The graph which is not connected will have 2 or more connected components.



- In a graph two vertices are said to be adjacent if there exist a ~~path~~ edge betn the two vertices.
- Two edges are said to be adjacent if there is a common vertex for the two edges.

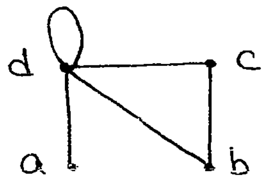
NULL GRAPH :-

- The graph with no edges is called Null graph / Empty graph

TRIVIAL GRAPH :-

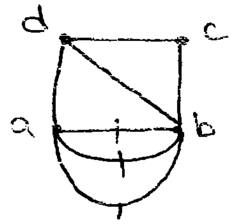
- The graph with one vertex and no edges.

Loop :- An edge is drawn from a vertex to itself.

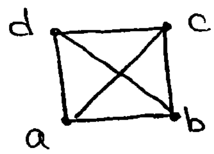


Parallel edges (Multiple edges) :-

- In a graph if a pair of vertex is allow to join by more than 1 edge then those edges are called parallel edges and resulting graph is called a Multigraph.



Simple Graph :- The graph with no loops and no parallel edges is called Simple Graph.



- Maximum no. of edges possible in a simple graph with n vertices $= \frac{n(n-1)}{2}$

$$^6C_6 + {^6C_5} + {^6C_4} + {^6C_3} + {^6C_2} + {^6C_1} + {^6C_0} = 2^6$$

- No. of simple graphs possible with n vertices $= 2^{n(n-1)/2}$

Ex : Max. no. of ~~edges~~ simple graphs possible ~~in a simple graph~~ with 5 vertices and 3 edges is ___

$\rightarrow$ Max. no. edges $= C(5,2) = 10$ $\left\{ \because \frac{n(n-1)}{2} = \frac{10 \times 2}{2} = 10 \right\}$

Req. no. Graphs $= C(10,3)$

$= 120$

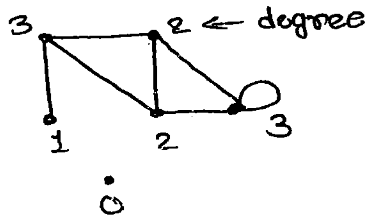
* No. of simple graphs possible with n vertices and m edges

$$= {}^{\frac{n(n-1)}{2}}C_m$$

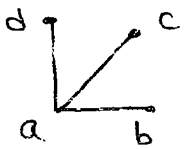
$$= C\left(\frac{n(n-1)}{2}, m\right)$$

* Degree of a vertex v = deg(v) = no. of edges incident with v

- In nondirected graph the loop at vertex is counted as two edges by writting degree of that vertex

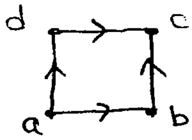


* In a simple graph with n vertices, Degree of any vertex
  $v = \deg(v) \le (n-1) \quad \forall\ v \in G$



simple graph

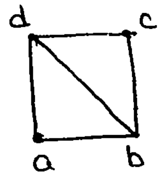
* In directed graph,
  indegree of a vertex is
  $\deg^{+}(v)$ = no. of edges incident to the vertex



  $\deg^{+}(d) = 1$

  outdegree = $\deg^{-}(v)$
  = no. of edges incident from the vertex

  $\deg^{-}(v) = \deg^{-}(d) = 1$



* $\delta(G)$ = Minimum of the degrees of all vertices in G.

  $\delta(G) = \{2, 3, 2, 3\}$
  $= 2$

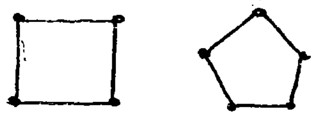
* $\Delta(G)$ = Max. of the degrees of all vertices in G

  $\Delta(G) = 3$

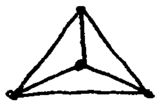
* Degree Sequence :-
- If the degrees of all vertices in G are arranged in descending order then the seq. to obtained is called Degree seq. of the G.

  D.S. = {3, 3, 2, 2}

* Regular Graph :-
- If all the vertices have same degree then the graph is called as Regular Graph.
- If the degree of each vertex is k then the graph is called as k-Regular graph.
- Every polygon is two Regular Graph.



2-regular



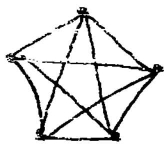
3-regular

<u>Complete</u> <u>Graph</u> :- In a simple graph with n mutually adjacent vertices is called complete graph.

— denoted by 'K_n'.

Ex :

K_2 K_3 K_4 K_4



K_5

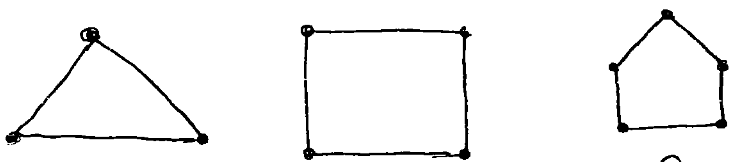
— Every complete graph is a regular graph but every regular graph need not be complete.

— A complete graph is a simple graph with max. no. of edges.

— No. of edges in K_n = $\frac{n(n-1)}{2}$

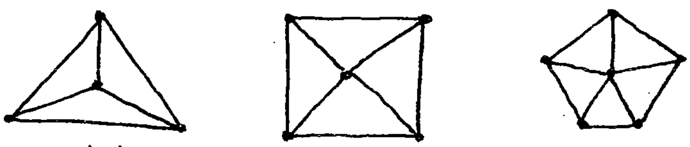
— Degree of each vertex = $(n-1)$

• <u>Cycle</u> <u>Graph</u> :- A simple graph G with n vertices $(n \geq 3)$ and n edges, is called a cycle graph, if all the edges from cycle of length n.



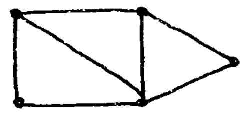
C_3 C_4 C_5

<u>Wheel</u> <u>Graph</u> :- A wheel graph with n vertices $(n \geq 4)$ can be formed from a cycle graph C_{n-1}, by adding a new vertex (hub) which is adjacent to all vertices of C_{n-1}

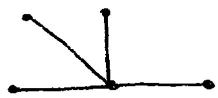


- Number of edges in $W_n = 2(n-1)$

Cyclic Graph :- A simple graph with atleast one cycle is called a cyclic graph.



Acyclic Graph :- A simple graph with no cycles is called Acyclic graph.

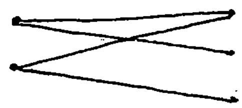


Tree : A connected graph with no cycle is called a tree

* A tree with n vertices has $(n-1)$ edges.
* Every tree has atleast two vertrces of degree 1.

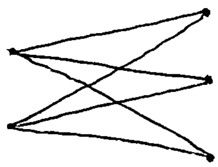
Bipartite Graph :- A simple graph $G = (V, E)$ with vertex partition $\{V_1, V_2\}$ is said to be a Biparotite graph, if every edge of G join a vertex in V_1 to a vertex in V_2.

$V = \{a, b, c, d, e\}$



Complete Bipartite Graph :- A bipartite graph $G = \{V, E\}$ with vertex partition $\{V_1, V_2\}$ is said to be complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2.

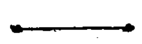
- If $|V_1| = m$ and $|V_2| = n$ then a complete bipartite graph from V_1 to V_2 is denoted by $K_{m,n}$



$K_{2,3}$

Note :- In general, A complete bipartite graph is not a complete graph.

- $K_{m,n}$ is a complete graph $\Leftrightarrow m = n = 1$



$K_{1,1}$ & K_2 (complete graph as well as complete bipartite graph)

- $K_{m,n}$ has $(m+n)$ vertices and mn edges.
- Max. no. of edges possible in bipartite graph with n vertices is

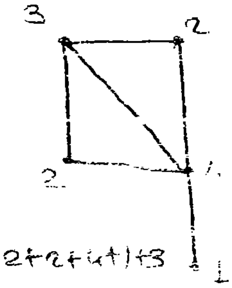
$n = 9$

$K_{5,4} \rightarrow 20$ $\quad \left\lfloor \frac{81}{4} \right\rfloor$

$K_{6,3} \rightarrow 18$ $\quad = \underline{\underline{20}}$

Sum of degrees theorem :-

Let $G = (V, E)$ be a non directed graph with $V = \{v_1, v_2, --- v_n\}$ then $\sum_{i=1}^{n} \deg(v_i) = 2 \cdot |E|$



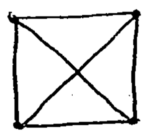
$\deg(v) = 2 \times 6$
$= 12$

$= 2+2+4+1+3$
$= 12$

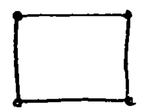
Corollary 1 : Let $G = (V, E)$ be a directed graph with $V = \{v_1, v_2 --- v_n\}$ then

$$\sum_{i=1}^{n} \deg^{+}(v_i) = \sum_{i=1}^{n} \deg^{-}(v_i) = |E|$$

Corollary 2 : In a nondirected graph, no. of vertices with odd degree is always even no. of vertices.



no. of vertices with odd degree = 4 (Even)



no. of vertices with odd degree = 0 (even)

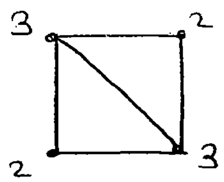
Corollary 3 : If G is a nondirected graph with degree of each vertex k, then $k \cdot |V| = 2|E|$

Corollary 4 : If G is a non directional graph with degree of each vertex atleast k ($\geq k$) then

$k \cdot |V| \leq 2|E|$

$\therefore 2 \times 4 \leq 2 \times 5$

$8 \leq 10$

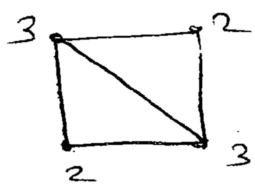


$k = 2$

Corollary 5 : If G is a non directed graph with degree of each vertex atmost k ($\leq k$) then

$k \cdot |V| \geq 2|E|$

$3 \cdot 4 \geq 2 \times 5$



$k = 3$

Note :- for a nondirected graph G

$$\delta(G) \cdot |V| \le 2|E| \le \Delta(G)|V|$$

$\uparrow$ Min. degree $\qquad \uparrow$ Max. degree.

Ex : A simple non directed graph G contains 21 edges, 3 vertices of degree 4, and other vertices are of degree 2. then $|V| = ?$

$\rightarrow$ $\sum_{i=1}^{n} deg(v_i) = 2|E| = 2 \times 21 = 42$

$$\Longrightarrow \quad 3 \times 4 + (x-3) \times 2 = 42$$

$$2x - 6 = 30$$

$$2x = 36$$

$$\boxed{x = 18}$$

$\therefore$ no. of vertices $= |V| = 18$

Ex : A simple non directed graph G has 34 edges and degrees of each vertex is 4 then find the $|V| = ?$

$\rightarrow$ $\sum_{i=1}^{n} deg(v_i) = 2 \times 34$

$$4 \times |V| = 68$$

$$|V| = \frac{68}{4}$$

$$\boxed{|V| = 17}$$

Ex : A simple non directed graph G has 24 edges and degree of each vertex is k then ~~xorz?~~ which of the following is possible no. of vertices ?

(a) 20 (b) 15 (c) 10 (d) 8

$\rightarrow$ By ~~cotto~~ corollary 3

$$k \cdot |V| = 2|E|$$

$$|V| = \frac{2 \times 24}{k} \qquad (k = 1, 2, 3, 4, 6)$$

$$= \frac{48}{} = \frac{48}{6} = 8 \qquad (48, 24, 16, 12, 8) \text{ in options}$$

Ex : Maximum no. of vertices possible in a simple non directed graph with 35 edges and degree of each vertex is atleast 3 is —

→ By cor. 4

$$k \cdot |V| \leq 2|E|$$

$$\Rightarrow 3 \cdot |V| \leq 2 \cdot 35$$

$$|V| \leq 23.33$$

$$|V| \leq 23$$

Ex : Min. no. of vertices of edges necessary in a simple non-directed graph, with 25 vertices and degree of each vertex is atleast 4 is —

→ By cor. 4

$$k \cdot |V| \leq 2|E|$$

$$4|25| \leq 2|E|$$

$$100 \leq 2|E|$$

$$50 \leq |E|$$

$$\therefore \boxed{E = 50}$$

For $k = 5$ (atleast 5)

$$k|V| \leq 2|E|$$

$$5|25| \leq 2|E|$$

$$|E| \geq 62.5$$

$$\boxed{|E| \geq 63}$$

Ex : Min. no. of vertices necessary in a simple non directed graph with 13 edges and degree of each vertex is atmost 3 is —

≤ 3

→ By cor. 5

$$k \cdot |V| \geq 2|E|$$

$$\Rightarrow 3 \cdot |V| \geq 2|13|$$

$$3 \cdot |V| \geq 26$$

$$|V| \geq \frac{26}{3} \Rightarrow |V| \geq 8.66$$

$$\boxed{|V| \geq 9}$$

Ex: Which of the following degree sequences represent a simple non directed graph?

a) $\{2, 3, 3, 4, 4, 5\}$ → sum of degrees $= 21 \neq$ Even

(OR)

no. of vertices with odd degrees $= 3$ (not even)

∴ not represent a simple non-directed graph.

b) $\{2, 3, 4, 4, 5\}$ → sum of degrees $= 18 =$ even

and no. of vertices with odd degrees $= 2 =$ Even

But it having 5 vertices and degree of Each vertex must be ≤ 4.

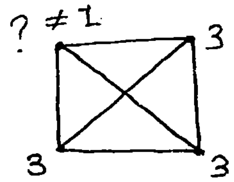
∴ It is not simple non directed graph.

c) $\{3, 3, 3, 1\}$ → sum of degrees $= 10 =$ Even

and no. of vertices with odd degree $= 4 =$ even

But if 3 vertices having 3 degree then remaining vertex cannot have degree 1

∴ It is not simple graph.



d) $\{1, 3, 3, 4, 5, 6, 6\}$ → sum of degree $= 28$

and no. of vertices with odd degree $= 4$

But it cannot represent simple non directed graph because in simple graph with 7 vertex it have two vertices with degree 6 then vertex with degree 1 is not possible.

e) $\{0, 1, 2, \ldots n-1\}$

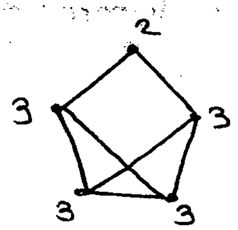
$\underbrace{v_1, v_2, v_3 \ldots v_n}$

Simple graph with n vertices can not represent simple graph because a simple graph with n vertices we have vertex with degree $n-1$ then vertex with degree 0 is not possible.

* In a simple graph with ~~$(n \geq 2)$~~ n vertices at least two vertices should have same degree.

f) $\{2, 3, 3, 3, 3\}$
a b c d e

→ Simple Graph.



Havel - Hakimi result

Consider the degree sequences S_1 and S_2, and assume that S_1 is in descending order.

S_1) $\{S, t_1, t_2, \ldots t_S, d_1, d_2, \ldots, d_n\}$

S_2) $\{t_1 - 1, t_2 - 1, \ldots, t_S - 1, d_1, d_2, \ldots d_n\}$

S_1 is graphic $\Leftrightarrow$ S_2 is graphic

Which of the Following degree sequences represent a simple non directed graph ?

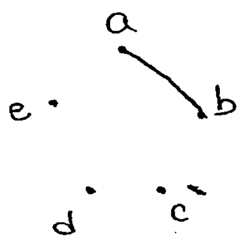
S_1) $\{6, \underline{6, 6, 6, 4, 3, 3}, 0\}$

$\{5, \underline{5, 5, 3, 2, 2}, 0\}$

$\{4, \underline{4, 2, 1, 1}, 0\}$

$\{3, \underline{1, 0, 0}, 0\}$

a b c d e



The last seq. can not be represented by simple graph.

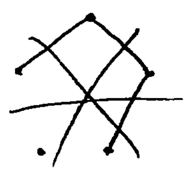
∴ Given seq. also can not be represented by a simple graph.

S_2) $\{6, \underline{5, 5, 4, 3, 3, 2}, 2, 2\}$

$\{\cancel{4}, \underline{4, 3, 2, 2}, 1, 2, 2\}$

$\{3, \underline{2, 1, 1}, 1, 2, 2\}$

$\{1, 0, 0, 1, 2, 2\}$



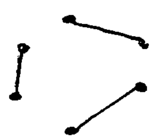
$= \{4, 4, 3, 2, 2, 1, 2, 2\}$

$= \{4, \underline{4, 3, 2, 2}, 2, 2, 1\}$ Reordering the Elements.

$= \{3, 2, 1, 1, 2, 2, 1\}$

$= \{3, \underline{2, 2, 2}, 1, 1, 1\}$

$\{1, 1, 1, 1, 1, 1\}$



$= \{1,1,1,1,0\}$

$= \{0,1,1,0\}$

$= \{1,1,0,0\}$

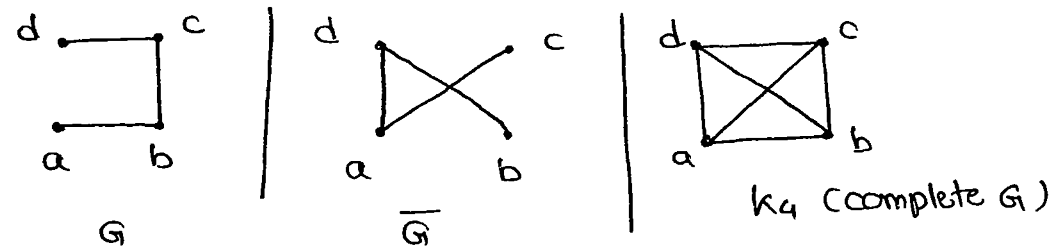
$= \{0,0,0\}$

$\therefore$ Given Graph is simple.

Complement of a graph :-

Let G be a simple non directed graph with n vertices. complement of G, denoted by $\overline{G}$ is a simple non directed graph with same vertices as that of G, and an edge $\{u,v\} \in \overline{G}$ iff $\{u,v\} \notin G$

Ex :



$G \cup \overline{G} = K_n$ — where n = no. of vertices.

$* \; |E(G)| + |E(\overline{G})| = |E(K_n)|$ — where $n = |V(G)|$

Ex : A simple graph G has 10 vertices and 21 edges find no. of edges in $\overline{G}$ $(|E(\overline{G})|)$.

$\rightarrow$ We have

$$|E(G)| + |E(\overline{G})| = |E(K_n)|$$

$$21 + |E(\overline{G})| = \frac{n(n-1)}{2} = \frac{10 \times 9}{2} = 45$$

$$|E(\overline{G})| = 45 - 21$$

$$|E(\overline{G})| = 24$$

Ex : A simple graph G has 30 edges and $\overline{G}$ has ~~25~~ 36 edges then no. of vertices in G = ?

$* \; n(n-1) = (12)(12-1)$

$\rightarrow$ $|E(G)| + |E(\overline{G})| = |E(K_n)|$

$$30 + 36 = \frac{n(n-1)}{2}$$

$\therefore \boxed{n = 12}$

Isomorphic Graphs :- Two Graphs G_1 & G_2 are said to be isomorphic, if there exists a function $f : V(G_1) \rightarrow V(G_2)$ such that i) f is a bijection and

ii) f preserves adjacency

i.e. if $\{u, v\} \in G_1$ then $\{f(u), f(v)\} \in G_2$ then $G_1 \equiv G_2$

Note :- If $G_1 \equiv G_2$, then

i) $|V(G_1)| = |V(G_2)|$

ii) $|E(G_1)| = |E(G_2)|$

iii) Degree sequences of G_1 & G_2 are same.

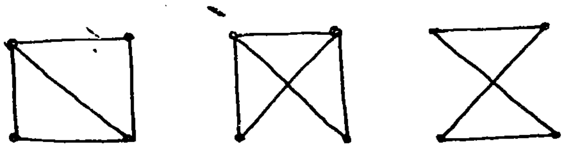
iv) If $\{v_1, v_2, --- v_k\}$ form a cycle in G_1, then $\{f(v_1), f(v_2), f(v_3), --- f(v_k)\}$ should form a cycle in G_2.

Result : $(G_1 \equiv G_2)$ iff $\overline{G_1} \equiv \overline{G_2}$

- $(G_1 \equiv G_2)$ if the adjacency matrices of G_1 and G_2 are same

- $(G_1 \equiv G_2)$ iff the corresponding subgraphs (obtained by deleting a vertex in G_1 and it's image in G_2) are isomorphic.

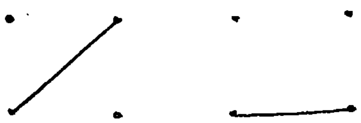
Ex : Which of the following graphs are isomorphic ?



G_1 $\qquad$ G_2 $\qquad$ G_3

→ Graph G_3 has only 4 edges $\therefore$ G_3 is not isomorphic to G_1 / G_2.

Now for G_1 & G_2

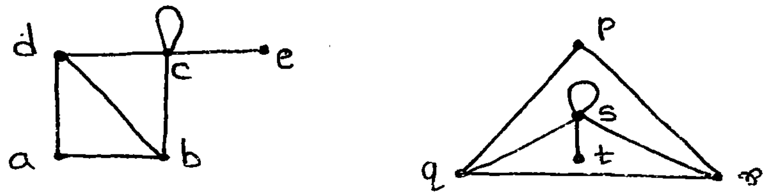


$\overline{G_1}$ $\qquad$ $\overline{G_2}$

$\overline{G_1} \equiv \overline{G_2}$

$\therefore G_1 \equiv G_2$

Ex : Find whether the following graphs are Isomorphic ?



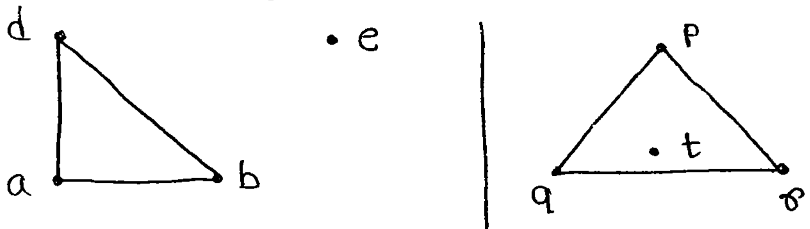
G_1 G_2

$\rightarrow$ If $G_1 \equiv G_2$,

Image of $c = s$

Further, c and s have similar neighbours.

Deleting c and s from G_1 & G_2 we get,

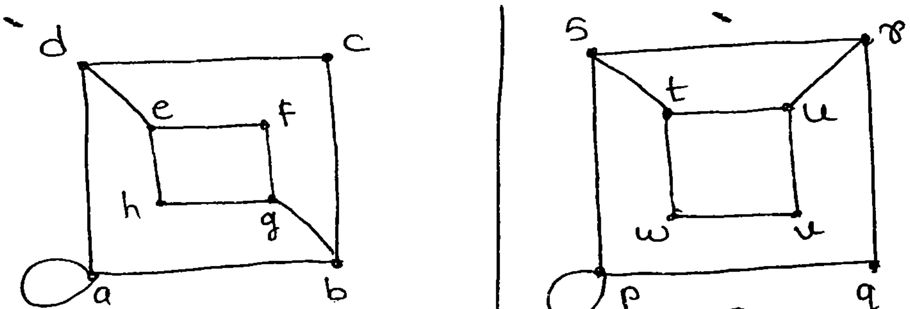


H_1 H_2

Here, $H_1 \equiv H_2$

$\therefore G_1 \equiv G_2$

Ex: Find whether the following graphs are isomorphic ?

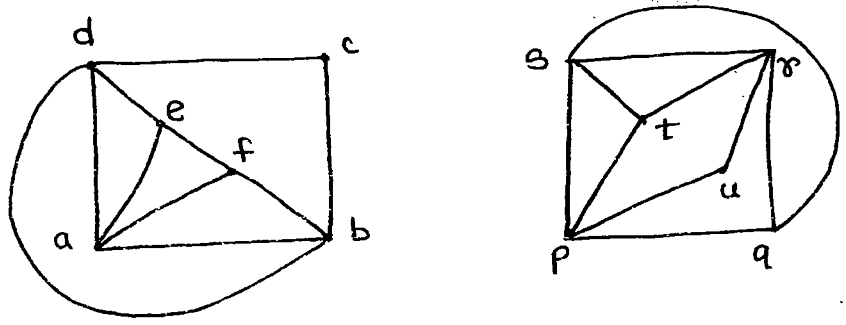


G_1 G_2

$\rightarrow$ In G_2 all vertices form the cycle C_8 where as in G_1 there is no such cycle

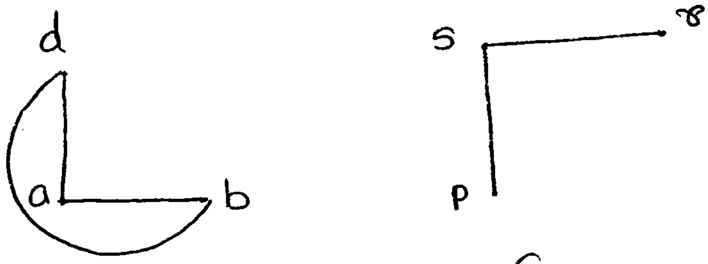
$\therefore G_1 \not\equiv G_2$

Ex: Find the wether Following graphs are isomorphic ?



G_1

→ consider vertices with degree 4



G_1 (Form cycle) G_2 (Not cycle)

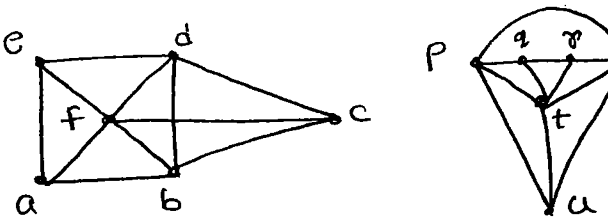
comparing vertices of degree 4 we have in G_1 vertices form a cycle but in G_2 not.

$\therefore G_1 \not\equiv G_2$

comparing vertices of degree 3 we have in G_1 two vertices of degree 3 are adjacent whereas in G_2 are not.

$\therefore G_1 \not\equiv G_2$

Ex: Find wheather the following graphs are isomorphic ?

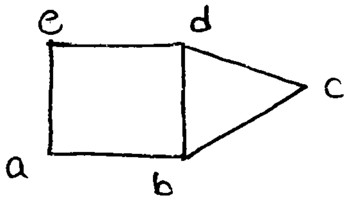


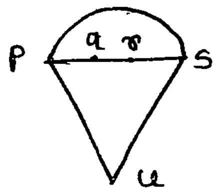
G_1 G_2

→ $G_1 \equiv G_2$ since Image of $f = t$

The vertices f & t has similar neighbours

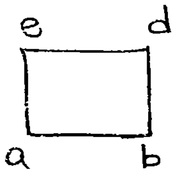
Deleting f & t from corresponding graphs we get following,

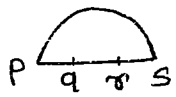




$G_{11} \equiv G_{21}$

Image of $c = u$ $\therefore$ Deleting c & u we get





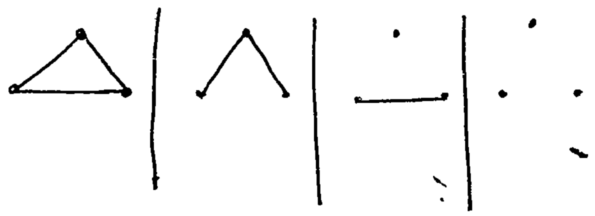
$G_{111} \equiv G_{211}$

$\therefore \; G_1 \equiv G_2$

Ex : How many simple non isomorphic graphs (pairwise) are possible with 3 vertices ?

a) 2 b) 3 ✓c) 4 d) 6

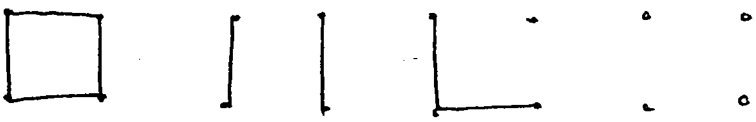
$\rightarrow$



Ex : How many simple non isomorphic graphs (pairwise) are possible with 4 vertices and 2 edges ?

✓a) 2 b) 3 c) 4 d) 6

$\rightarrow$



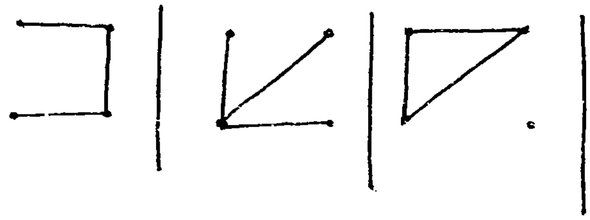
(2 - edges) (2 - edges)

Two edges may be adjacent or non adjacent.

Ex : —||— with 4 vertices and 3 edges ?

a) 2 b) 3 c) 4 d) 6

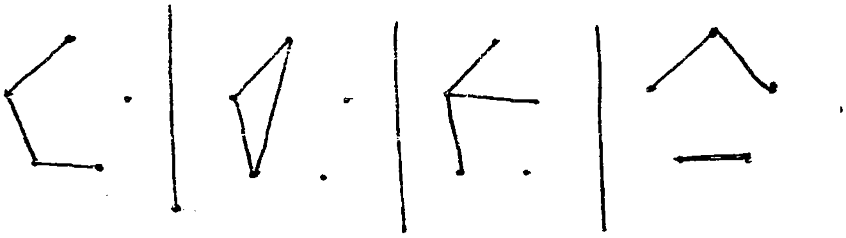
→



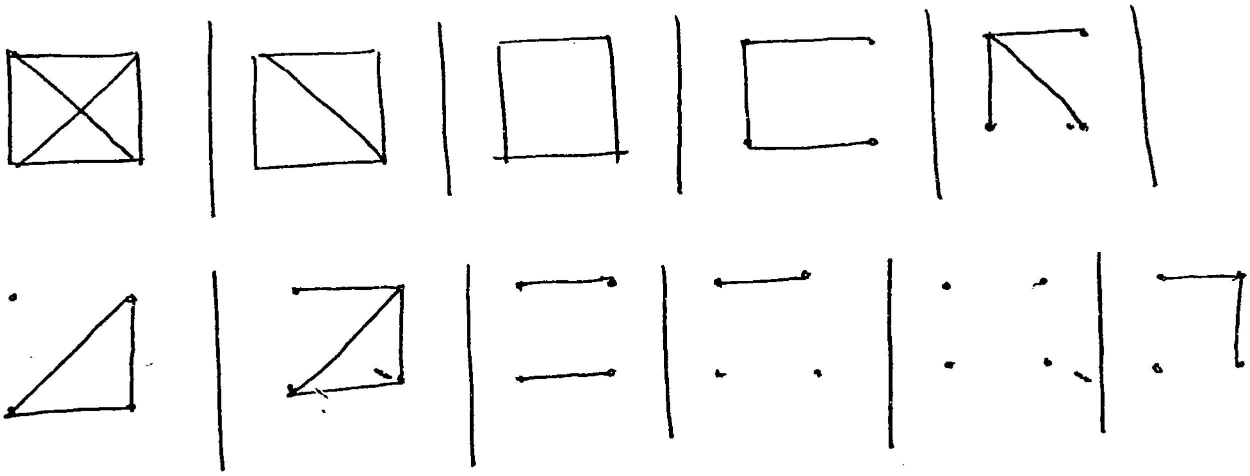
Ex : —||— with 5 vertices and 3 edges ?

a) 2 b) 3 c) 4 d) 6

→



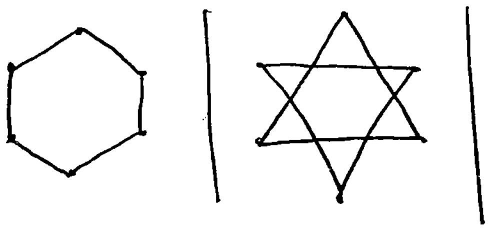
Ex : —||— 4 vertices ?



⇒ Ans = 11

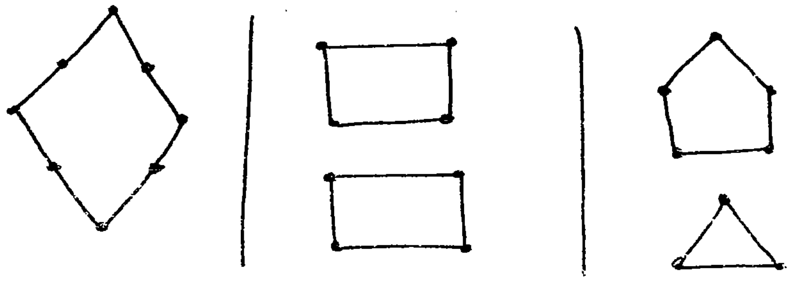
Ex : How many simple non isomorphic graphs (pairwise) are possible with 6 vertices, 6 edges and degree of each vertex is 2 ?

→



Ans : 2

Ex : ——"—— with 8 vertices , 8 edges and degree is 2 ?

→ 

Ans = 3

Ex . ——"—— with 10 vertices , 10 edges and degree is 2 ?

→

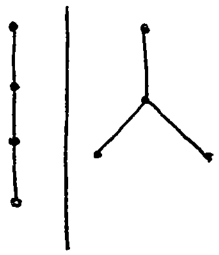
C_{10}	C_5	C_6	C_7	C_4
	C_5	C_4	C_3	C_3
				C_3

Ans = 5

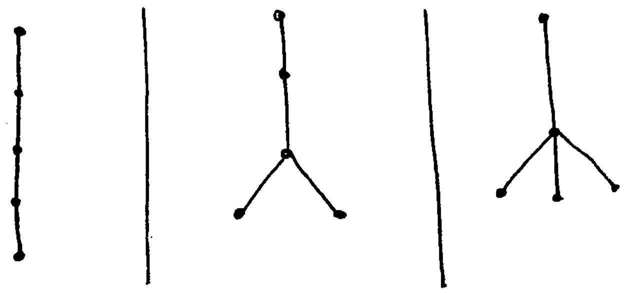
Ex : How many simple non isomorphic trees (pairwise) are possible with 3 vertices ?

→  Ans = 1

Ex : ——"—— with 4 vertices ?

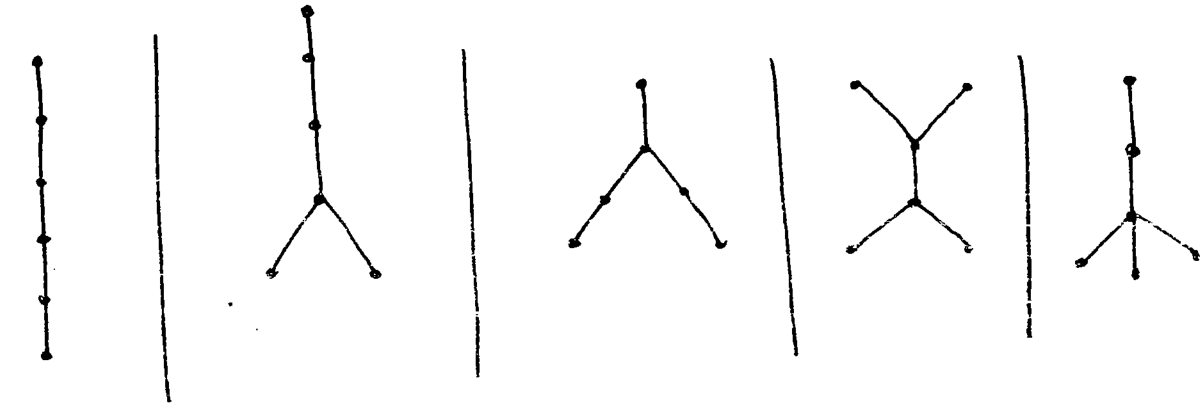
→  Ans : 2

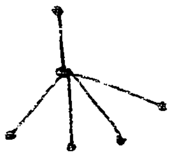
Ex : ——"—— with 5 vertices ?

→  Ans = 3

Ex : —— || —— with 6 vertices ?

→

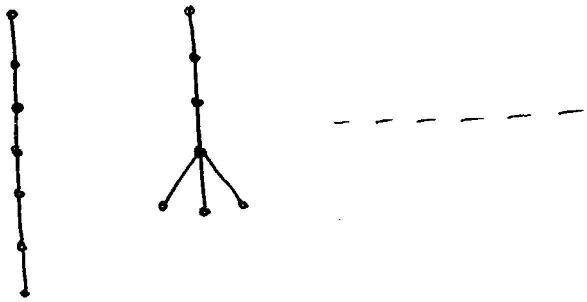




Ans = 6

Ex : —— || —— with 7 vertices ? Ans = 12

→



* If a simple graph G is isomorphic to $\overline{G}$ then

i) $|E(G)| = \frac{n(n-1)}{4}$ where n = no. of vertices in a graph

$|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2}$

$\therefore\ 2|E(G)| = \frac{n(n-1)}{2}$ $\left(\because |E(G)| = |E(\overline{G})|\right)$

$\therefore\ |E(G)| = \frac{n(n-1)}{4}$

ii) $|V(G)| = 4k$ or $(4k+1)$ $k = 1, 2, 3 ----$

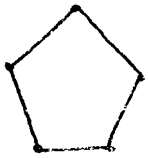
Ex : If $C_n \equiv \overline{C_n}$ then $n = ?$

$\rightarrow$ By the prev. result no. of edges in $C_n = \dfrac{n(n-1)}{4}$

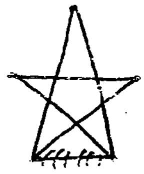
$$|E(C_n)| = \frac{n(n-1)}{4}$$

$$n = \frac{n(n-1)}{4}$$

$\boxed{n = 5}$



k_5



$\overline{k_5}$

Ex : If $G \equiv \overline{G}$ then which of the following is not true ?

a) $|V(G)| = 20$ b) $|V(G)| = 21$

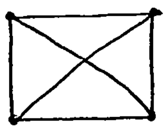
c) $|V(G)| = 25$ ~~d)~~ $|V(G)| = 26$

$\Rightarrow$ $|V| = 4k$ or $4k+1$

Planar Graphs

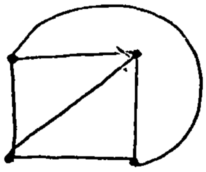
A graph G is said to be planar, if it can be drawn on a plane (at a non vertex point) so that no two edges cross each other

Ex :



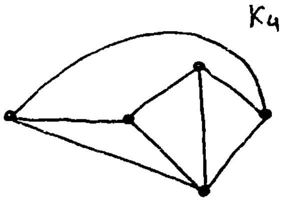
K_4

$\Longrightarrow$

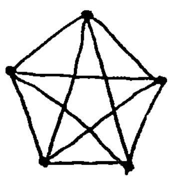


K_4

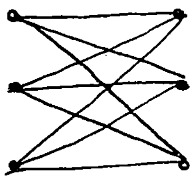
(Planar graph)



(Planar graph)



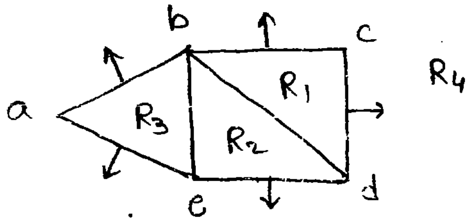
K_5



$K_{3,3}$

(non planar graphs)

Regions : Every planer graph divide the plane into connected areas called Regions of the plane.



$\deg(R_1) = \deg(R_2) = \deg(R_3) = 3$

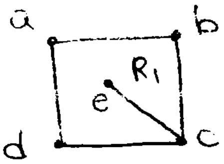
$\deg(R_4) = 5$

Degree of an interior region $r = \deg(r)$

= no. of edges enclosing the region r

Degree of exterior region $r = \deg(r)$

= no. of edges exposed to the region r.



$\deg(R_1) = 6$

Properties of Planar Graph :-

1) $\sum_{i=1}^{n} \deg(v_i) = 2|E|$

2) sum of degrees of regions theorm :-

→ If P is a planar graph with n regions

$$\sum_{i=1}^{n} \deg(r_i) = 2|E|$$

2.1) In a planar graph if degree of each region is k then

$k.|R| = 2|E|$

2.2) In a planar graph if degree of each region is $\geq k$ (atleast k) then $k.|R| \leq 2.|E|$

2.3) In a simple planar graph degree of each region ≥ 3 then

$3.|R| \leq 2.|E|$

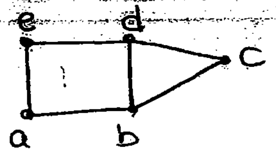
2.4) In a planar graph degree of each region $\leq k$ (atmost k) then $k.|R| \geq 2|E|$

3) Euler's formula :-

If G is a connected planar graph, then

$|V| + |R| = |E| + 2$

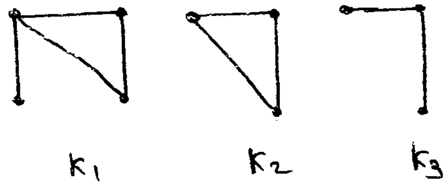
$5+3 = 8+2$



— If G is planer graph with k components

$|V| + |R| = |E| + (k+1)$

$10 + 3 = 9 + (3+1)$



4) If G is a simple connected planer graph, then

i) $|E| \le [3|V| - 6]$

ii) $|R| \le \{2|V| - 4\}$

iii) There exists atleast one vertex $v \in G$ such that $\deg(v) \le 5$

5) Polyhedral graph : A planer graph in which every interior region is polygon is called polyhedral graph.

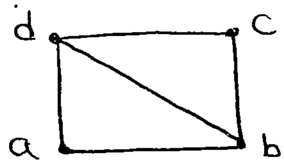
— In a polyhedral graph degree of every vertex i.e. $\deg(v) \geq 3$ $\forall\ v \in G$

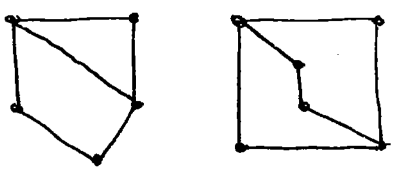
— For a polyhedral graph the following inequality must be holds :-

i) $3|V| \le 2|E|$ and ii) $3|R| \le 2|E|$

6) Kuratowski's theorem :- A graph G is non planer iff G has a subgraph is homeomorphic to K_5 or $K_{3,3}$.

Homeomorphic : Two graph G_1 & G_2 are said to be homeomorphic to each other if each of these graphs can be obtained from a graph G by dividing some edges of G with more vertices





G_1 G_2

Note :- 1) Any graph with 4 or fewer vertices is planar

2) Any graph with 8 or fewer edges is planar.

3) A non planar graph minimum no. of vertices is k_5

4) A non planar graph with min. no. of edges is $k_{3,3}$

5) A complete graph k_n is planar iff $n \leq 4$

6) $k_{m,n}$ is planar. iff ($m \leq 2$ or $n \leq 2$)

- If two graphs are isomorphic then they are homomorphic.

Ex. Let G be a connected planar graph with 25 vertices and 60 edges. No. of regions in G is __

→ By Euler's Formula,

$|V| + |R| = |E| + 2$

$\Rightarrow 25 + |R| = 60 + 2$

$\Rightarrow |R| = 37$

Ex: Let G be a planar graph with 10 vertices, 3 components and 9 edges then $|R| = ?$

→

$|V| + |R| = |E| + (k+1)$

$\Rightarrow 10 + |R| = 9 + (3+1)$

$\Rightarrow |R| = 3$

Ex: Let G be a connected planar graph with 20 vertices, degree of each vertex is 3 then $|R| = ?$

→ By sum of vertices of degrees theorem

$\sum deg(v_i) = 2|E|$

$20 \times 3 = 2|E|$

$\boxed{|E| = 30}$

$|V| + |R| = |E| + 2$

$|R| = 32 - 20$

$\boxed{|R| = 12}$

Ex: Let G be a connected planar graph with 35 regions, degree of each region is 6 then $|V| = ?$

$\rightarrow$ $\sum_{i=1}^{35} \deg(r_i) = 2|E|$

$35 \times 6 = 2|E|$

$|E| = 105$

$\therefore |V| + |R| = |E| + 2$

$|V| = 107 - 35$

$\boxed{|V| = 72}$

Ex: Let G be a connected planar graph with 12 vertices & 30 edges and degree of each region is k Then $k = ?$

$\rightarrow$

$|V| + |R| = |E| + 2$

$12 + |R| = 32$

$|R| = 20$

By sum of degrees of regions theorm

$20 \cdot k = 2|E|$

$k = \frac{60}{20}$

$\boxed{k = 3}$

Ex: Max. no. of regions possible in a simple planar graph with 10 edges is_

$\rightarrow$ By theorm 2.3

$3|R| \leq 2|E|$

$3|R| \leq 20$

$|R| \leq \frac{20}{3}$

$|R| \leq 6.66$

$\Rightarrow |R| \leq 6$

Ex : Min no. of edges necessary in a simple planar graph with 15 regions is —

→ By theom. 2.3

$3|R| \leq 2|E|$

$3|15| \leq 2|E|$

$|E| \geq 22.5$

$\Rightarrow |E| \geq 23$

Ex : Max. no. of edges possible in a simple connected planar graph with 8 vertices is —

→ By Theorm 4.1

$|E| \leq \{3|V| - 6\}$

$3 \cdot 8 - 6$

$|E| \leq 18$

Ex : Max. no of regions possible in simple planar connected graph with 13 vertices is —

→ By theorm 4.2

$|R| \leq \{2|V| - 4\}$

$= \{2 \times 13 - 4$

$\Longrightarrow |R| \leq 22$

Ex : If G is simple connected planar graph, The $\delta(G)$ cannot be equal to —

↓ Min. degree of all vertices

→ a) 3 b) 4 c) 5 d) 6

By theorm 4.3

$V \in G$ such that

$deg(V) \leq 5$

$\Rightarrow \delta(G) \leq 5$

$\Rightarrow \delta(G) \neq 6$

Ex: Which of the following is /are true ?

S_1) A polyhedral graph with 30 edges and 11 regions does not exists.

→ By Euler's formula

$|V| + |R| = |E| + 2$

$|V| + 11 = 30 + 2$

$|V| = 21$

from polyhedral graph

$3|V| \leq 2|E|$ and $3|R| \leq 2|E|$

⇒ $3 \times 21 \leq 2(30)$ $3 \times 11 \leq 2 \times 30$

$63 \leq 60$ $33 \leq 60$

∴ True.

S_2) A polyhedral graph with 7 edges does not exist

→ $3|V| \leq 2|E|$ and $3|R| \leq 2|E|$

$|V| \leq 2 \times 7/3$ $|R| \leq 4.66$

$|V| \leq 4.66$ $|R| \leq 4$

$|V| \leq 4$

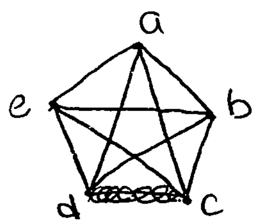
∴ By Euler's formula ⇒ $|V| + |R| = |E| + 2$

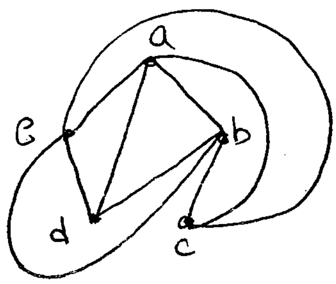
$4 + 4 = 7 + 2$

$8 \neq 9$ $(8 \ngtr 9)$

∴ True

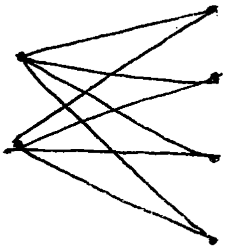
Ex: Which of the following is not a planar graph ?

a) 

⇒ 

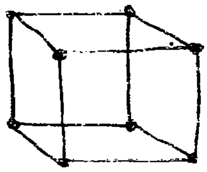
Planar Graph

b)

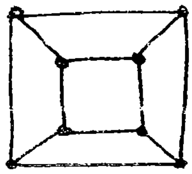


Planar

c)

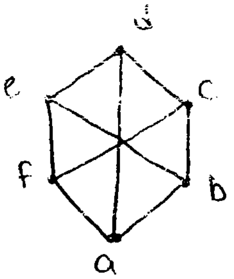


≡



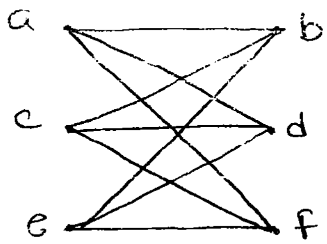
planar

d)



non planar

≡



$K_{3,3}$

Ex: G is a non planar graph with minimum no. of vertices then

a) G has 6 vertices and 8 edges
b) G has 5 —ıı— and 9 —ıı—
c) G has 6 —ıı— and 9 —ıı—
~~d)~~ G has 5 —ıı— and 10 —ıı— (K_5)

Ex: Let G be a non planar graph with minimum no. of edges then —

a) G has 6 vertices and 8 edges
b) G has 5 —ıı— and 9 —ıı—
~~c)~~ G has 6 —ıı— and 9 —ıı— ($K_{3,3}$)
d) G has 5 —ıı— and 10 —ıı—

Graph Coloring :-

Vertex coloring :- An assignment of colors to the vertices of a graph G, so that no two adjacent vertices have same color is called vertex coloring.

Chromatic Number : Min. no. of colors required for vertex coloring of a graph G is called as chromatic number and denoted by

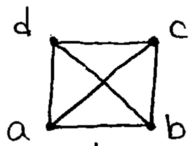
$G = \chi(G)$

* $\chi(G) = 1$ iff G is a null graph.
* If G is not a null graph, Then $\chi(G) \geq 2$
* A graph G is said to be n-colorable, if there exist a vertex coloring that uses atmost n colors i.e. $\chi(G) \leq n$
* Four color theorm
  - Every planar graph G is 4-colorable i.e. $\chi(G) \leq 4$
* Welch - Powel's algorithm
  1) Arrange vertices of G in the descending order of their degrees.
  2) If 2 or more vertices of same degree, arrange those vertices in Alphabetical or numerical order. ~~so that~~
  3) Assign colors to the vertices in that order so that no 2 adjacent vertices have same color.

Ex: Chromatic number of the complete graph K_n is —

a) n b) $n-1$ c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→



K_4

In K_n each vertex is adjacent to remaining $(n-1)$ vertices

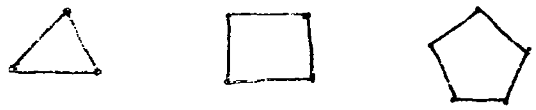
∴ Each vertex requires a new color.

$\therefore \chi(K_n) = n$

Ex : Chromatic no. of the graph C_n is ___
(cycle)

a) 2 b) 3 c) $n - 2\lfloor n/2 \rfloor + 2$ d) $n - 2\lceil n/2 \rceil + 1$

→



If n is odd then 3
If n is even then 2 } verify option for even & odd

Ex : Chromatic no. of the wheel graph W_n $(n \geq 4)$ is ---

→

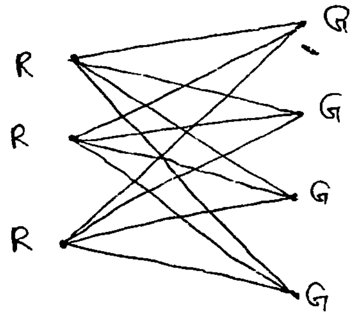
W_7 W_4

If n is even 4
If n is odd 3

Ex : chromatic no. of the complete bipartite graph $K_{m,n}$ is ___

a) 2 b) 4 c) min (m,n) d) max of {m,n}

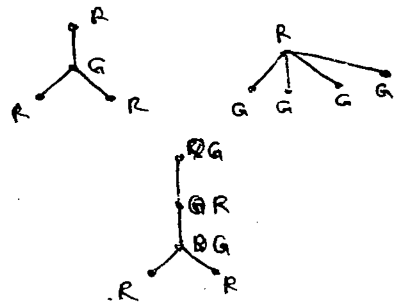
→



Ex : chromatic no. of a tree T with n vertices $(n \geq 2)$ is ___

a) 2 b) 4 c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→

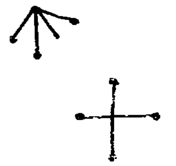


Every tree is bipartite graph
$\therefore \chi(n) = 2$

Ex : chromatic no. of the star graph with n vertices $(n \geq 2)$ is —

~~a)~~ 2 b) 4 c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→

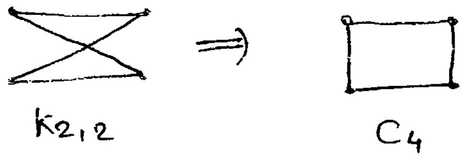


Star graph is special case of bipartite graph. ↓ $K_{1,n-1}$

$\therefore \chi(n) = 2$

Ex : In a graph G if all the cycles are of even length then $\chi(n) = ?$

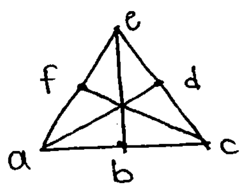
→ when cycle of even length then it is bipartite graph.



$\chi(n) = 2$

Ex : chromatic no. of the graph shown below is —

~~a)~~ 2 b) 3 c) 4 d) 5



vertex	a	b	c	d	e	f
color	C_1	C_2	C_1	C_2	C_1	C_2

$\chi(G) \leq 2$ — ①

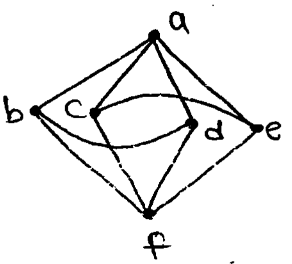
As G is not null graph

$\therefore \chi(G) \geq 2$

From ① & ②

$\chi(G) = 2$

Ex: chromatic no. of the graph shown below is



a) 2 b) 3 c) 4 d) 5

vertex	a	b	c	d	e	f
color	C_1	C_2	C_2	C_3	C_3	C_1

$\chi(G) \leq 3$ —— ①

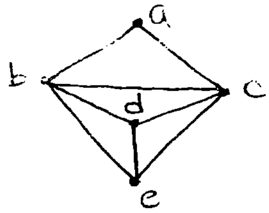
further, we have 3 mutually adjacent vertices $\{a, b, d\}$

$\therefore \chi(G) \geq 3$ —— ②

From ① & ②

$\chi(G) = 3$

Ex: chromatic no. of the graph given below is ——



a) 2
b) 3
c) 4
d) 5

vertex	a	b	c	d	e
color	C_1	C_2	C_3	C_1	C_4

Here graph is planar therefore by 4 color theorem chromatic no. $\chi(G) \leq 4$ —— ①

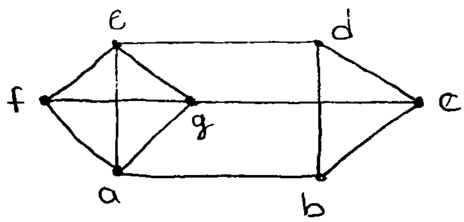
Further we have 4 mutally adjacent vertices $\{b, c, d, e\}$

$\therefore \chi(G) \geq 4$ —— ②

from ① & ②

$\chi(G) = 4$

Ex:



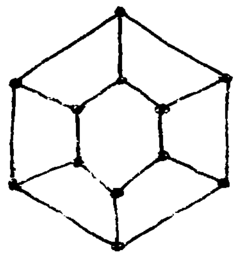
(Planar Graph)

a) 2
b) 3
c) 4
d) 5

vertex	a	b	c	d	e	F	g
color	C_1	C_2	C_1	C_4	C_2	C_3	C_4

Ex : chromatic no. of the graph shown below is —

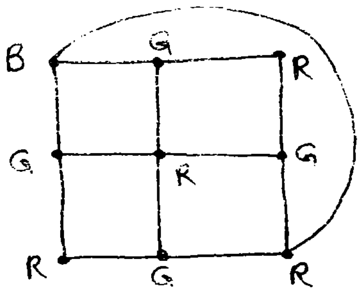
a) 2 b) 3 c) 4 d) 5



In the graph all cycles are of even vertices

$\therefore \chi(G) = 2$

Ex :

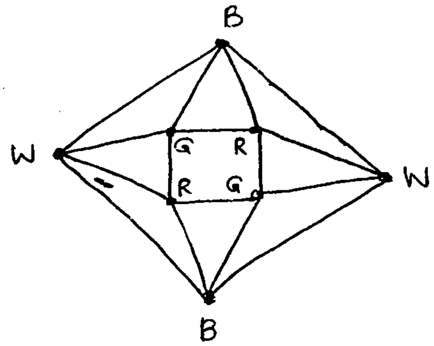


a) 2
b) 3
c) 4
d) 5

The graph we have a cycle of odd length

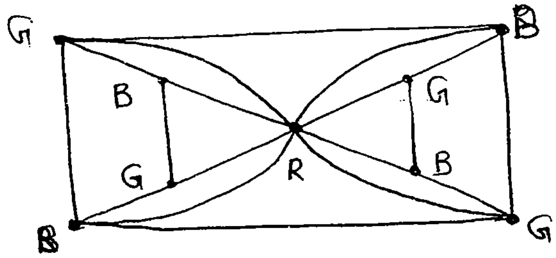
$\therefore \chi(G) \geq 3$

Ex : Chromatic no. of graph shown Below is —



a) 2
b) 3
c) 4
d) 5

Ex : chromatic no. of the graph shown below is —

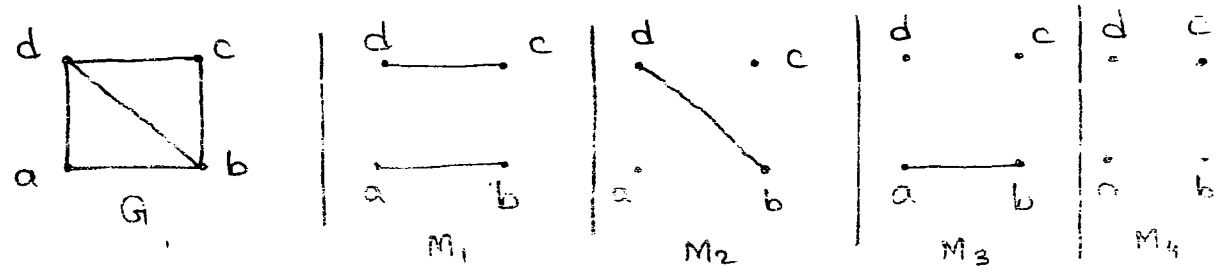


a) 2
b) 3
c) 4
d) 5

Matching and Covering

Matching : Let G be a graph. A subgraph M of G is called a matching of G, if every vertex of G is incident with almost one vertex in M.

i.e. $deg(v) \leq 1 \quad \forall\, v \in G$



- In a matching, no two edges are adjacent

$deg(v) \leq 1$

- **Maximal Matching** : A matching M of a graph G is said to be maximal, if no other edges of G can be added to M.

  - In above example for the graph G M_1 & M_2 are maximal matching of G.

- **Maximum Matching** : A matching of a graph with maximum no. of (largest maximal matching) edges is called a maximum matching of G.

  For a graph given in above example M_1 is maximum matching

- **Matching Number** : The no. of edges in a maximum matching of G.

  Matching no. of $M_1 = 2$

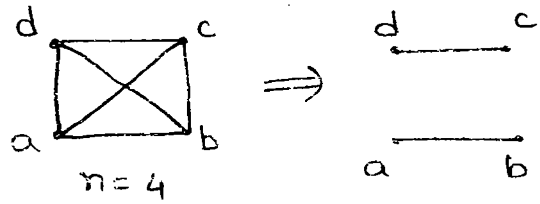
- **Perfect Matching** : A matching of a graph in which every vertex is matched is called perfect match.

- **Note** :- Every Perfect Matching is Maximum Matching / Largest maximal matching. (not vice versa)
  - Converse of above need not be true.
  - If Graph G has a perfect match then no. of vertices in G is even. (but not vice versa)

Ex : Matching number of K_n ~~(n is even)~~ is ___

a) $\lfloor n/2 \rfloor$ b) $\lceil n/2 \rceil$ c) $\lfloor \frac{n+1}{2} \rfloor$ d) $\lceil \frac{n+1}{2} \rceil$

→



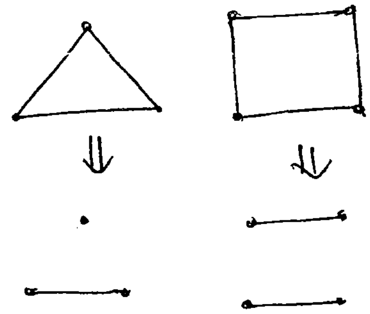
$= n/2$ if n is even

$= \frac{n-1}{2}$, if n is odd

Ex : What is matching no. C_n $(n \geq 3)$ is ___

a) $\lfloor n/2 \rfloor$ b) $\lceil n/2 \rceil$ c) $\lfloor \frac{n+1}{2} \rfloor$ d) $\lceil \frac{n+1}{2} \rceil$

→

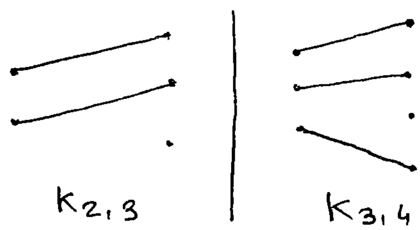


Ex : what is matching no. of $K_{m,n}$ is —

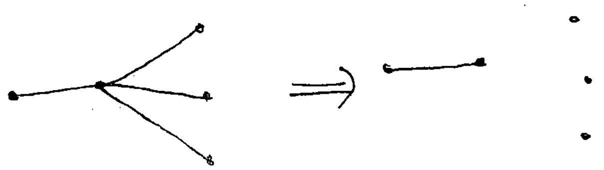
a) Max of $\{m, n\}$ b) min of $\{m, n\}$

c) $|m-n|$ d) G.C.D of $\{m, n\}$

→



Ex : what is the matching no. of star graph with n vertices $(n \geq 2)$ is

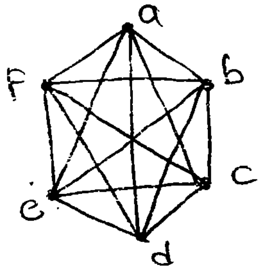


Star graph is $K_{1, n-1}$ Bipartite graph

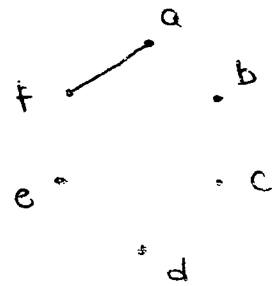
Ans : 1

* K_n has a perfect matching iff n is even.

Ex: How many no. of perfect matching in K_{2n} is ____



$a \to 5$ matches
$b \to 3$ –"–
$e \to 1$ –"–



$\Rightarrow$ 15 matching no.

Generally,
$V_1 \to (2n-1)$
$V_2 \to (2n-3)$
$V_3 \to (2n-5)$
$\vdots \quad\quad \vdots$

$\therefore$ total perfect matching no. $= (2n-1)(2n-3)(2n-5)\cdots$

$$= \frac{\lfloor 2n}{2n(2n-2)(2n-4)\cdots}$$

$$= \frac{\lfloor 2n}{2^n \lfloor n}$$

Ex: How many perfect matching in K_8 is ____

$\to \quad 2n = 8$

$$\therefore \text{no.} = \frac{8!}{2^4\, 4!} = \frac{\cancel{8}\times 7\times \cancel{6}^{3}\times 5}{\cancel{16}\,\cancel{2}} = 105$$

$\Rightarrow$ $K_{m,n}$ has a perfect matching iff $m=n$.

Ex: No. of perfect matching in $K_{n,n}$ $(n \geq 1)$ is ____

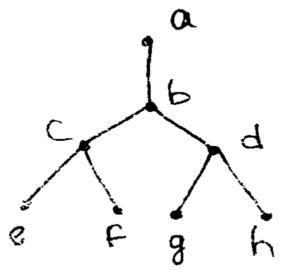
n way $\leftarrow$	V_1	u_1
$(n-1) \leftarrow$	V_2	u_2
$(n-2) \leftarrow$	V_3	u_3
	V_4	u_4
	$\vdots$	$\vdots$
$1 \leftarrow$	V_n	u_n

$$= n(n-1)(n-2)\cdots 1$$

$$= \underline{\lfloor n}$$

* A tree can have atmost 1 perfect matching.

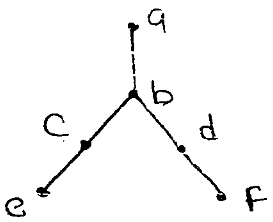
Ex: No. of perfect matching in the tree shown below is ___



a) 0
b) 1
c) 2
d) 3

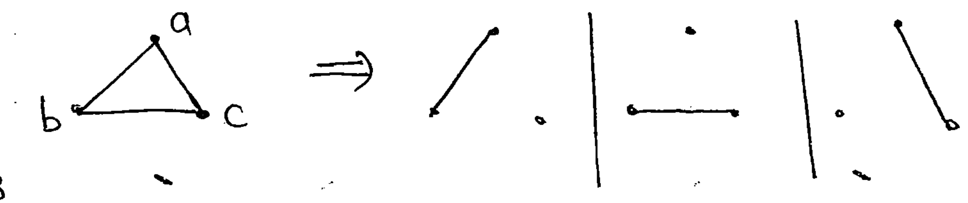
In a perfect matching degree of every vertex is 1 therefore, we cannot delete any edge passing through leaf nodes.

Ex: No of perfect matching in the tree given below is ___



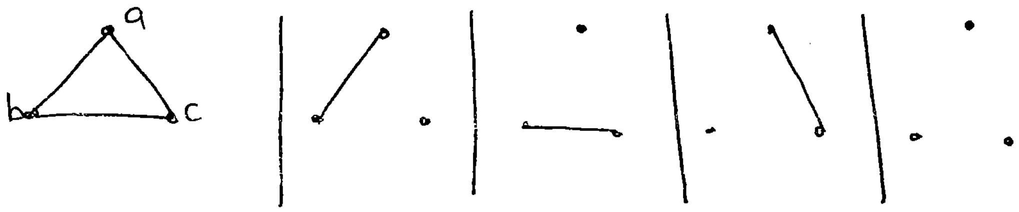
a) 0
b) 1
c) 2
d) 3

Ex: No. of maximal matching in the graph shown below is ___



Ans = 3

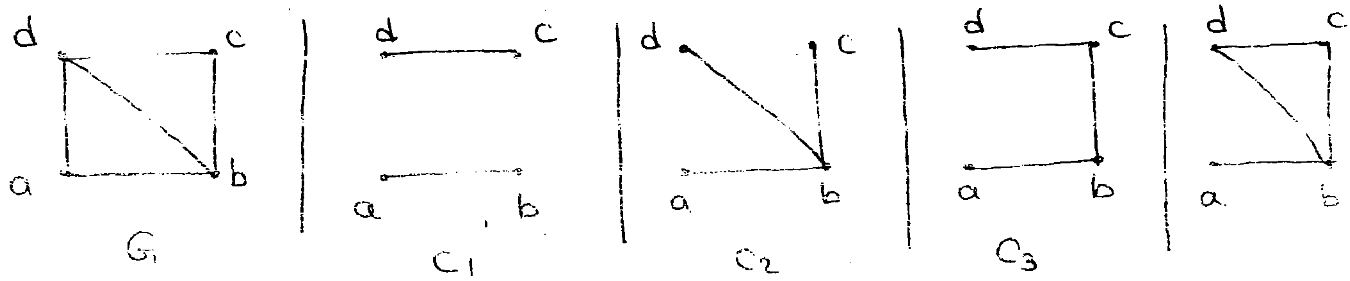
Ex: No. of matching in the graph shown below is 4



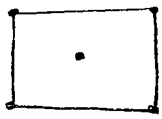
Coverings :-

Line Covering :- Let $G = (V, E)$ be a graph. A subset C of E is called a line covering of G, if every vertex of G is incident with atleast one edge in C.

i.e. $\deg(v) \geq 1 \quad \forall v \in G$



— line covering of a graph G does not exists iff G has an isolated vertex.



Minimal line covering : A line covering is said to be minimal, if no edge can be deleted from the line covering, without destroying it's ability to cover the graph.

For the graph given in the above example C_1 & C_2 are minimal line covering.

Minimum Line covering : (smallest minimal line covering)

A line covering with minimum no. of edges is called a minimum line covering.

* The no. of edges in minimum line covering is called line covering number of a graph. $G = \alpha_1$

The graph is above example has $\alpha_1 = 2$.

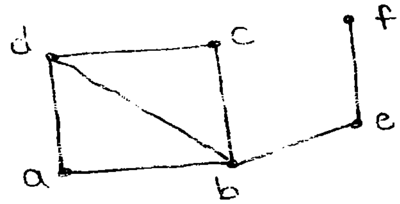
(C_1 graph is minimum line covering.)

* line covering of graph with n vertices contains atleast $\lceil \frac{n}{2} \rceil$ edges.
* No minimal line covering can contain a cycle.
* In a line covering if there is no path of length 3 or more, then C is minimal.

* In the line covering if there are no path of length 3 or more then all components of C is star graph. Then from star graph no edge can be deleted.

<u>Independent line set</u> :

Let $G = (V, E)$ be a graph. A subset L of E is called an independent if no two edges in L are adjacent.



$L_1 = \{(b,d)\}$

$L_2 = \{(b,d), (e,f)\}$

$L_3 = \{(a,d), (b,c), (e,f)\}$

$L_4 = \{(a,b), (e,f)\}$

<u>Maximal independent line set</u> : An independent line set L of a graph G is said to be maximal, if no other edges of G can be added to L.

Ex : $L_2 = \{(b,d), (e,f)\}$

$L_3 = \{(a,d), (b,c), (e,f)\}$ are maximal independent line set because no other edges can be added to it.

<u>Maximum independent line set</u> :- (Largest Maximal independent line set) An independent line set L of a graph G, with maximum no. of edges is called Maximum independent line set.

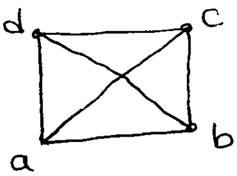
* No. of edges in Maximum independent line set is called line independent number of G denoted by β_1 (say)

* line independent no. = Matching no. of G.

For the graph in above example L_3 is a Maximum independent line set and $\beta_1 = 3$.

* For any graph G, $\boxed{\alpha_1 + \beta_1 = |V|}$

Ex : For the graph shown below which of the following is a minimal line covering ?



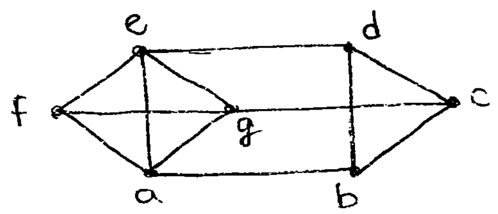
(a) $\{(a,b), (a,c), (b,d)\}$ (c) $\{(a,b), (a,c), (a,d)\}$

(b) $\{(a,b), (b,d), (d,c)\}$ (d) $\{(a,b), (c,d), (a,c)\}$

Ex : No. of edges in the line covering of the graph cannot exceed (with n vertices)

a) $\lceil \frac{n}{2} \rceil$ b) $\lfloor n/2 \rfloor$ c) $n-1$ d) $n-2$

Ex : For the graph shown below



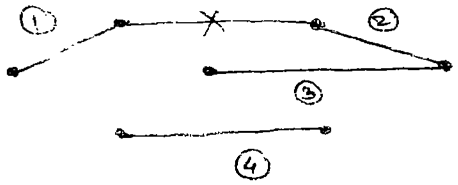
line covering no. $= \alpha_1 = 4$

line independent no. $= \beta_1 = 3$

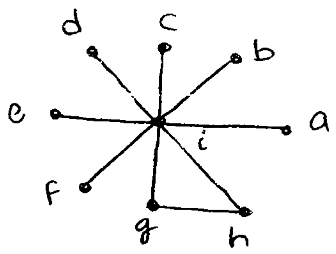
$\beta_1 = |V| - \alpha_1$

$= 7 - 4$

$= 3$



Ex : For the graph shown below



$\alpha_1 = 7$

$\alpha_1 + \beta_1 = |V|$

$\beta_1 = 9 - 7$

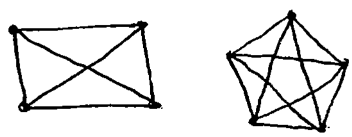
$\beta_1 = 2$

Ex : For the complete graph K_n find α_1, β_1

$\alpha_1 = \lceil n/2 \rceil$

$\beta_1 = \lfloor n/2 \rfloor$

$\alpha_1 + \beta_1 = |V| = n$



Ex : For the cycle graph C_n $(n \geq 3)$ α_1 & $\beta_1 = ?$

$\alpha_1 = \lceil n/2 \rceil$

$\beta_1 = \lfloor n/2 \rfloor$

$\alpha_1 + \beta_1 = n$

Ex : wheel graph W_n $(n \geq 4)$

$\alpha_1 = \lceil n/2 \rceil$

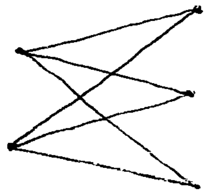
$\beta_1 = \lfloor n/2 \rfloor$

Ex: For complete bipartite graph $K_{m,n}$

$\alpha_1 = \max(m,n)$

$\beta_1 = \min(m,n)$

$\alpha_1 + \beta_1 = m+n$



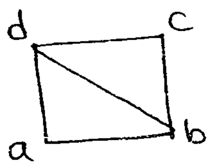
Ex: For the star graph with n vertices $(n \geq 2)$

$\alpha_1 = n-1$

$\beta_1 = 1$

$\alpha_1 + \beta_1 = n$

Vertex Covering :- Let $G = (V,E)$ be graph. A subset k of V is called a vertex covering of G, if every edge of G is incident with a vertex in k.



$k_1 = \{b, d\}$

$k_2 = \{a, b, c\}$

$k_3 = \{b, c, d\}$

Minimal Vertex Covering :- vertex covering k of a graph G is said to be minimal if no vertex can be deleted from k.

k_1 & k_2 are minimal vertex covering.

Minimum vertex covering :- A vertex covering of a graph G with minimum number of vertices is called as Minimum vertex covering. (also called as smallest minimal vertex covering)

– No. of vertices in a minimum vertex covering is called vertex covering no. of Graph G, denoted by α_2

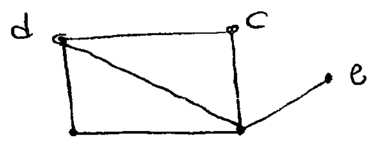
In above graph G of k_1 $\alpha_2 = 2$

Independent Vertex set :-

Let $G = (V,E)$ be a graph. A subset S of V is called an independent set if no two vertices in S are adjacent.

$S_1 = \{b\}$ $\quad$ $S_4 = \{a, c\}$

$S_2 = \{d, e\}$



Maximal indep. vertex set :- An independent vertex set is said to be maximumal, if no other vertex of G can be added to the set.

Ex : $S_1 = \{b\}$

$S_2 = \{d, e\}$

$S_3 = \{a, c, e\}$

Maximum indep. vertex set (Largest maximal indep. vertex set) :-

An indep. vertex set of graph G with maximum no. of vertices is called Maximum indep. vertex set.

* The no. of vertices in Maximum indep. vertex set is called as ~~indep~~. vertex indep. number of G denoted by β_2

Ex : $S_3 = \{a, c, e\}$

$\therefore \beta_2 = 3$

* For any graph $\alpha_2 + \beta_2 = |V|$

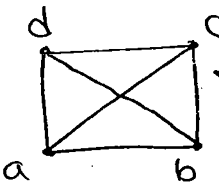
* For any graph if S is independent set of G then $V - S$ = A vertex covering of G.

Ex : For a complete graph K_n, vertex covering no $= \alpha_2 = ?$
vertex indep. no $= \beta_2 = ?$

$\longrightarrow$ $\alpha_2 + \beta_2 = |V| = n$

$\beta_2 = 1$

(because in K_n every vertex adjacent to each other)



$S_1 = \{a\}$

$\therefore \alpha_2 = |V| - \beta_2$

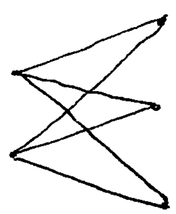
$\alpha_2 = n - 1$

Ex : For the complete bipartite graph $K_{m,n}$

$\alpha_2 = \min(m, n)$

$\beta_2 = \max(m, n)$

$\alpha_2 + \beta_2 = m + n$



Ex: for the star graph with n vertices $(n \geq 2)$

$\rightarrow$ $\alpha_2 = 1$

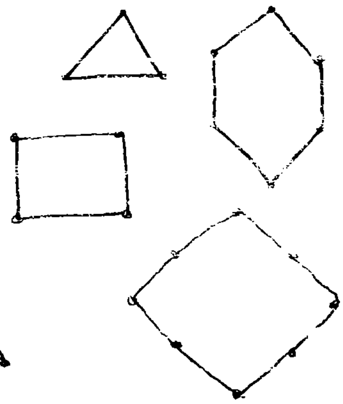
$\beta_2 = n-1$

$\alpha_2 + \beta_2 = n$

Ex: For the cycle graph C_n $(n \geq 3)$

$\alpha_2 = \lceil n/2 \rceil$

$\beta_2 = \lfloor n/2 \rfloor$

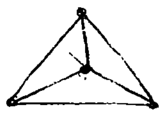


Ex: wheel graph W_n $(n \geq 4)$

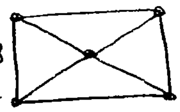
$\rightarrow$ $\alpha_2 = \left\lceil \frac{n+1}{2} \right\rceil$

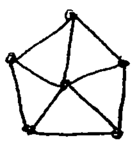
$\beta_2 = \left\lfloor \frac{n-1}{2} \right\rfloor$

$\alpha_2 = 3$
$\beta_2 = 1$



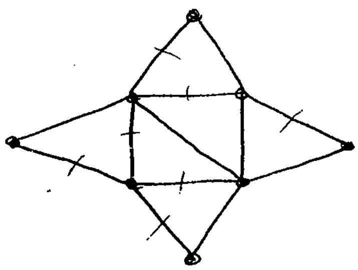
$\alpha_2 = 3$
$\beta_2 = 2$





$\alpha_2 = 4$
$\beta_2 = 2$

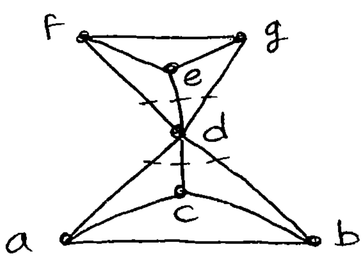
Ex: For the graph shown below



$\beta_2 = 4$

$\alpha_2 = 4$

Ex:

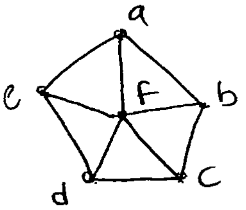


which is not true?

a) $\alpha_2 = 5$ c) $\{a, e\}$ is a max. indep. set

b) $\beta_2 = 2$ d) $\{a, c, e, f\}$ is vertex covering

Ex:



which of the following is not true?

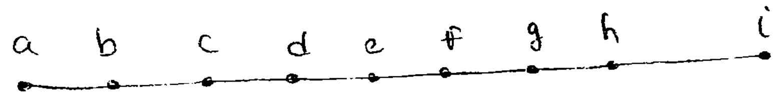
a) $\alpha_2 = 4$ b) $\beta = 2$

c) $\{b, d\}$ is max. indep. vertex set

d) $\{a, c, f\}$ is a vertex covering.

Ex : The no. of vertices in the smallest maximal indep. vertex set in the chain of 9 nodes :

(a) 3   b) 4   c) 5   d) 2



$S_1 = \{a, c, e, g, i\}$
$S_2 = \{b, d, f, h\}$ } Largest Maximal

$S_3 = \{b, e, h\} \longrightarrow$ smallest

Spanning Trees

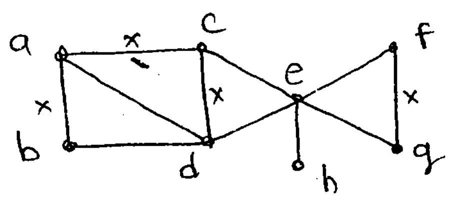
Let G be a connected graph. A subgraph H of G is called a spanning tree of G, if i) H is a tree

ii) H contains all vertices of G

Circuit rank :- Let G be connected graph with n vertices and m edges. Any spanning tree of G contains (n-1) edges

- The no. edges we have to delete from G to get spanning tree of G is equal to $\boxed{m-(n-1)}$ is called circuit rank of G.

Ex :



Circuit rank (G) = m-(n-1)   (m → Edges)

= 11 - (8-1)

= 4

Ex : Let G be a connected graph with 6 vertices and degree of each vertex is 3 circuit rank of G = ?

→

$\Sigma \deg(V_i) = 2|E|$

$18 = 2|E|$

$|E| = 9$

Circuit rank (G) = 9 - (n-1)

= 9 - (6-1)

= 4

Ex: Let G be connected graph with 5 vertices, max. no. of edges and all cycles in G are of even length. Find $C(G) = ?$

→ If all cycles of even length then G is bipartite graph with max. no. edges and 5 vertices.

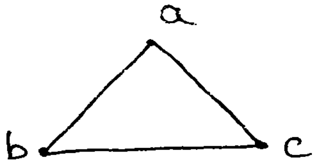
$\therefore G = K_{3,2}$ or $K_{2,3}$

$\therefore$ No. edges $= 6$

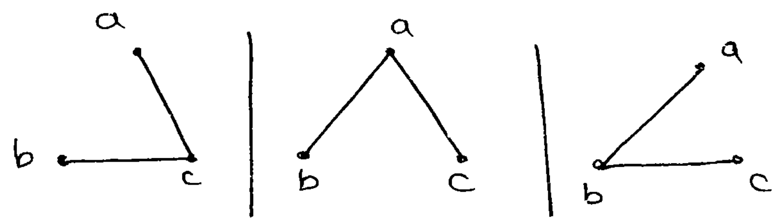
$C(G) = 6 - (5-1)$

$= 2$

Ex. Number of spanning trees in the graph shown below is —

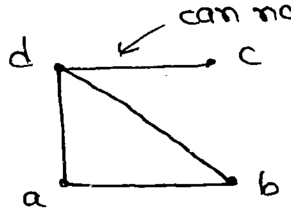


a) 3 ✓
b) 4
c) 6
d) 8



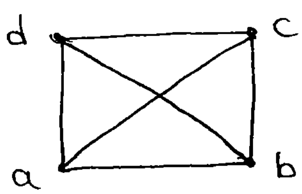
Ex: No. of spanning trees —

can not delete



a) 3 ✓
b) 4
c) 6
d) 8

Ex: No. of spanning trees —



a) 6
b) 8
c) 12
d) 16 ✓

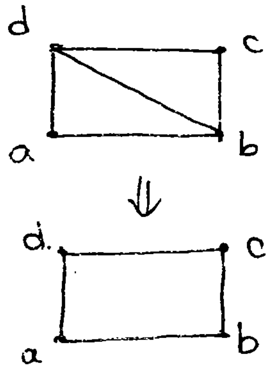
No. of spanning trees in complete graph $= n^{n-2}$ (Caley's Formula)

$= 4^{4-2}$

$= 16$

Circuit rank $= m - (n-1) = 6 - (4-1) = 3$

Ex : No. of spanning trees —



a) 6
b) 8
c) 10
d) 4

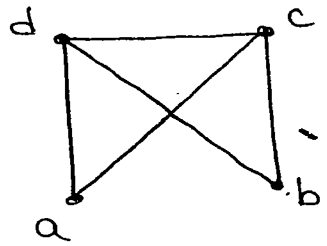
Kirchoff's Theorm :-

Let A be the adjacency matrix of a connected graph G. Let M be the matrix obtained from A, by changing all 1's into −1 and replacing each zero element in the principle diagonal of A with the degree of corresponding vertex.

— cofactor of any element of M is equal to the no. of spanning trees in G.

Ex : No. of spanning trees in the graph shown below :-

$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



$$M = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

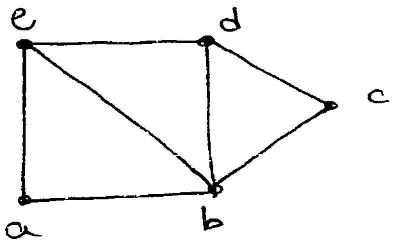
$$\text{cofactor of } M_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 2(9 \oplus -1) + 1(-2) - 1(1+3)$$

$$= 16 - 2 - 4 - 2$$

$$= 8$$

Ex : No of spanning trees in the graph shown below :-

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



$$M = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

$$\text{cofactor of } M_{51} = (-1)^{5+1} \begin{vmatrix} -1 & 0 & 0 & -1 \\ 4 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \end{vmatrix}$$

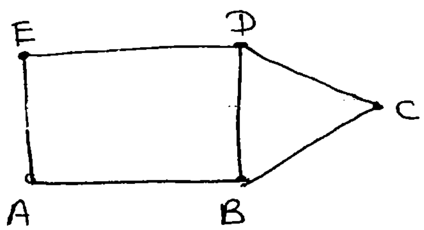
$C_4 - C_1$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 4 & -1 & -1 & -5 \\ -1 & 2 & -1 & 1 \\ -1 & -1 & 3 & 0 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} -1 & -1 & -5 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{vmatrix} = 21$$

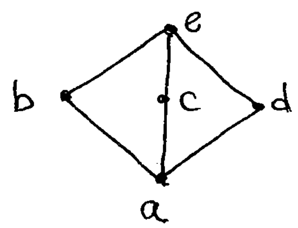
Ex : No. of spanning trees —

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Ans = 11

Ex :



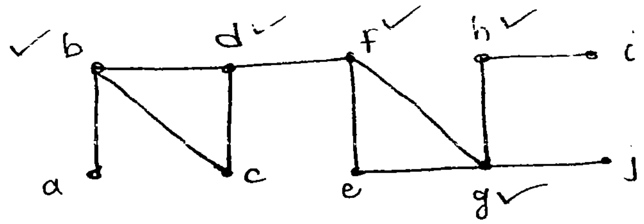
Ans :- 12

IW BFS, DFS, Minimal spanning trees, Kruskal's algo, Prim's Algorithm, Traversability theorm, Euler circuit, Hamiltonian graph.

Connectivity :- A graph is not connected will have two or more connected components.

Cut Vertex :- Let G be a connected graph. A vertex $v \in G$ is called cut vertex of G if $(G-v)$ results in a disconnected graph.

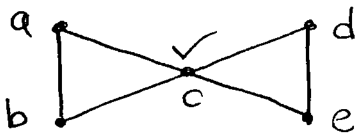
(Articulation point)



Cut edge :- Let G be a connected graph and an edge $E \in G$ is called a cut edge of G if $(G-E)$ results in a disconnected graph.

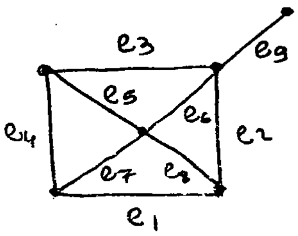
(Bridge)

- In a connected graph G if $E \in G$ is cut edge iff E is not part of any cycle in G.
- whenever cut edge exists cut vertex also exists and not vice versa. Because atleast one vertex of cut edge is a cut vertex.
- If cut vertex exists then cut edge may or may not be exists. Ex: vertex c,



Cut set :- Let $G = (V, E)$ be a connected graph. A subset E' of E is called a cut set of G, ~~iff (G-E') results in a disconnected graph~~ If deletion of all the edges in E' makes G disconnected and deletion of no proper subset of E' from G, can make G disconnected.

Ex: For the graph shown below which of the following are cut set ?



✓ (a) $\{e_1, e_3, e_5, e_7\}$

× (b) $\{e_2, e_5, e_6, e_8\}$

× (c) $\{e_2, e_6, e_3, e_9\}$

↓

No need of e_9.

because using e_2, e_6, e_3 it is disconnected graph.

✓ (d) $\{e_9\}$

✓ (e) $\{e_1, e_4, e_7\}$

<u>Edge connectivity</u> :- In a connected graph G, the min. no. of edges whose deletion makes G disconnected is called 'Edge connectivity of G'.

It is denoted by $\lambda(G)$.

- If G has a cut edge then edge connectivity $\lambda(G) = 1$
- The no. of edges in a smallest cut set of G is said to be edge connectivity of G.

<u>Vertex Connectivity</u> :- In a connected graph G, the min. no of vertices whose deletion makes graph disconnected or reduces G into a trivial graph is called vertex connectivity of a connected graph G.

It is denoted by $k(G)$

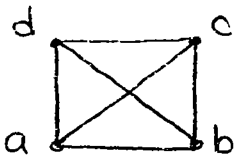
- If G has a cut vertex then $k(G) = 1$
- For any connected graph G,

$$\boxed{k(G) \leq \lambda(G) \leq \delta(G)}$$

$\delta(G)$ = min. of all the degrees of all vertices.

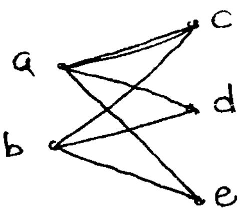
Ex: For complete graph K_n

vertex connectivity of $K_n = n-1$
Edge connectivity of $K_n = n-1$



Ex: For a complete bipartite graph $K_{m,n}$

vertex connectivity = min (m, n)
edge connectivity = min (m, n)

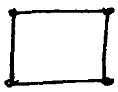


$\delta(G) = 2$

$k(G) \leq \lambda(G) \leq 2$

Ex: For a cycle graph C_n $(n \geq 3)$

Edge connectivity = 2
vertex connectivity = 2

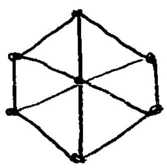


$\delta(G) = 2$

$k(G) \leq \lambda(G) \leq 2$

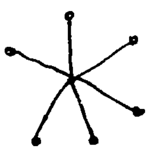
Ex: For the wheel graph W_n $(n \geq 4)$

Edge connectivity = 3
vertex connectivity = 3



$k(G) \leq \lambda(G) \leq \delta(G) = 3$

Ex: For the star graph $(n \geq 2)$ $(K_{1,n-1})$

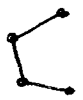


edge connectivity = 1
vertex connectivity = 1

$\delta(G) = 1$

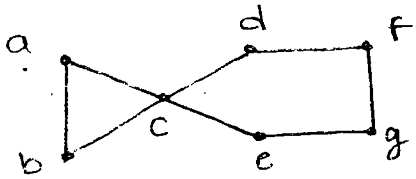
Ex: For a tree with n vertices ($n \geq 2$),

vertex connectivity = 1
edge connectivity = 1



$\delta(G) = 1$

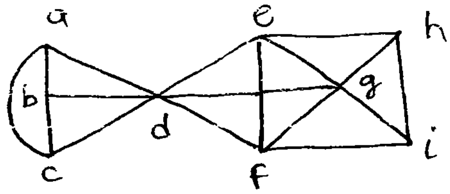
Ex: For the graph shown below



$\delta(G) = 2$

vertex connectivity = 1
edge connectivity = 2

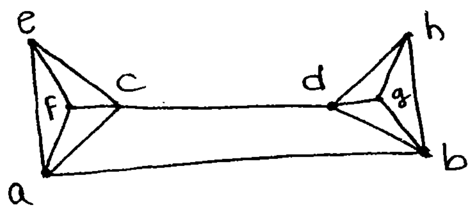
Ex:



$\delta(G) = 3$

vertex conn. = 1
Edge conn. = 3

Ex:



$\delta(G) = 3$
vertex conn. = 2
edge conn. = 2

* A simple graph G with n-vertices is necessarily connected if number of edges in G

$$\boxed{|E(G)| > \left\{\frac{(n-1)(n-2)}{2}\right\}}$$

Ex: Which of the following Graphs are necessarily connected?

(A) A Graph with 6 V and 10 E ($10 \not> 10$)
(B) 7 vertices and 14 edges ($14 \not> 15$)
(C) 8 vertices, 22 edges ($22 > 21$) ✓
(D) 9 vertices, 28 edges ($28 \not> 28$)

Ex: what is the min. no. of edges needed to guarantee the connectivity in a simple graph with 10 vertices?

(a) 37 b) 38 c) 34 d) 28

$|E(G)| > \frac{(n-1)(n-2)}{2} \Rightarrow 9 \times 4$
$= 36$

* A simple graph with n-vertices and k-components has atleast (n - k) edges.

$$i.e. \ |E(G)| \geq (n-k)$$

* A simple graph with n-vertices and k-components then the max. no. of edges of G is always at most $\frac{(n-k)(n-k+1)}{2}$ edges.

$$|E(G)| \leq \frac{(n-k)(n-k+1)}{2}$$

Ex : Min. no. of edges necessary in a simple graph with 10 V and 3 components.

a) 25 b) 28 c) 30 ✓d) 7

$\rightarrow$ $|E(G)| \geq (n-k)$

$\geq (10-3)$

$|E(G)| \geq 7$

Ex: Max. no. of edges necessary in a simple graph with 10 V and 3 components

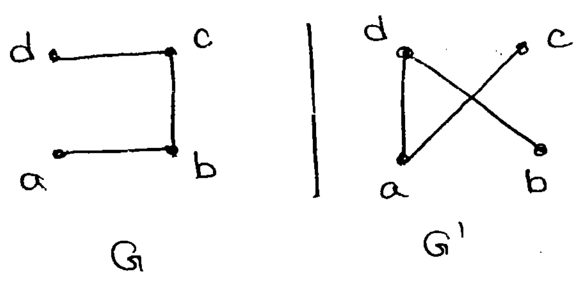
a) 25 ✓b) 28 c) 30 d) 37

$$|E(G)| \leq \left[\frac{(10-3)(10-3+1)}{2}\right]$$

$$|E(G)| \leq 28$$

Ex: Which of the following is/are true?

(a) If a simple graph G is connected then $\overline{G}$ is not connected.

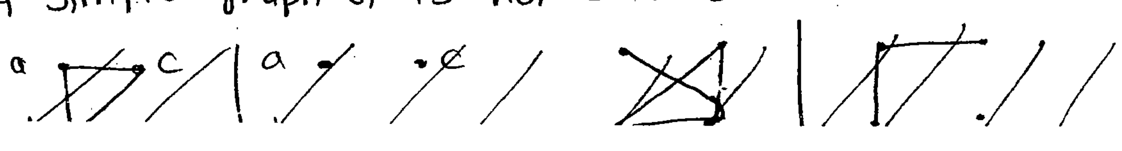


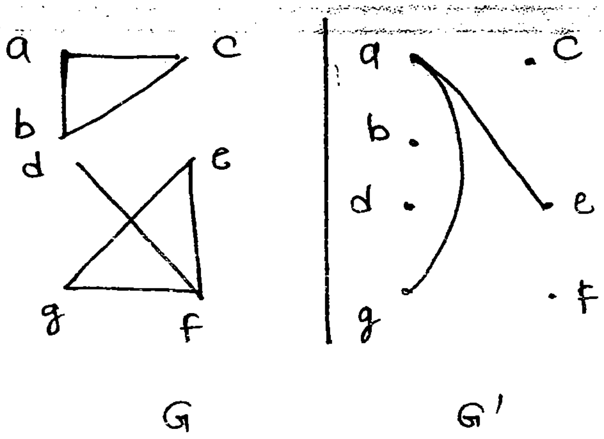
Here G & G' both connected.

$\therefore$ <u>FALSE</u> (Need not be always true)

TRUE

(b) A simple graph G is not connected then $\overline{G}$ is connected





TRUE

(c) In a complete graph G with n-vertices, if $\delta(G) \geq \left(\frac{n-1}{2}\right)$ then G is connected.

→ Suppose given stat. is False

i.e. $\delta(G) \geq \frac{(n-1)}{2}$ and G is not connected

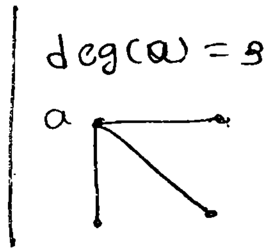
G has atleast two components G_1 & G_2

Let $v \in G_1$

$$deg(v) \geq \left(\frac{n-1}{2}\right)$$

$$\Rightarrow |V(G_1)| \geq \frac{n-1}{2} + 1$$

$$|V(G_1)| \geq \frac{n+1}{2}$$



Similarly, no. of vertices in G_2 is also $\geq \frac{n+1}{2}$

Now, No. of vertices in G $\geq |V(G_1)| + |V(G_2)|$

$$\geq \frac{n+1}{2} + \frac{n+1}{2}$$

$$|V(G)| \geq n+1$$

∴ It is contradiction.

∴ Given stat. is true.

TRUE

(d) If a simple graph G has exactly two vertices of odd degree, then there exists a path betn the 2 vertices.

→ The given stat. is true if G is connected graph.

If a graph is nonconnected then by sum of degree theorm the two vertices of odd degree belong to same components x: The G be a graph with n vertices and k components. If we delete a vertex in G, then the number of components in G, should lie betn

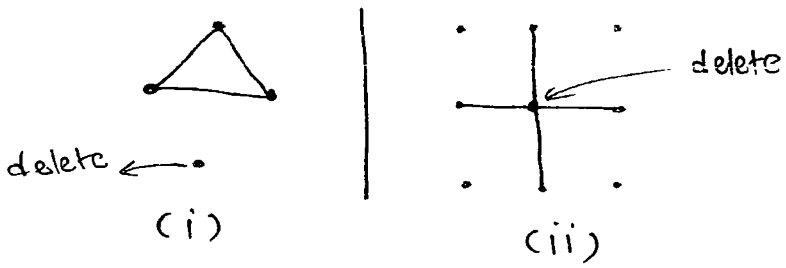
a) $n-k$ and $n-1$ b) $k-1$ and $n-k$

c) k and $n-k$ ~~(d)~~ $k-1$ and $n-1$

i) If the vertex we are deleting is the component of itself then the no. of components become $k-1$

ii) If given graph is star graph with n vertices then by deleting cut vertex of star graph we get $n-1$ components

∴ max. no. component = $n-1$

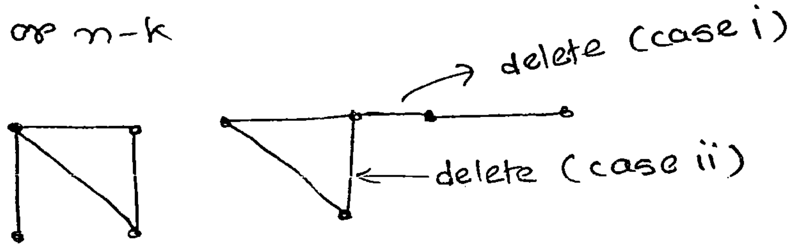


Ex: Let G be a graph with n vertices and k-components. If we delete an edge in G, then ~~the~~ no. of components in G are —

a) $k-1$ or k ~~(b)~~ k or $k+1$ c) $k-1$ or $k+1$

d) k or $n-k$

→



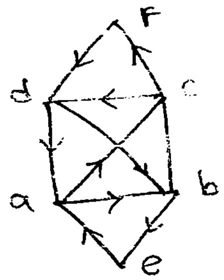
i) If edge we are deleting is cut edge then no. of components become $k+1$.

ii) If deleting edge is not cut edge then no. of components remains same i.e. k

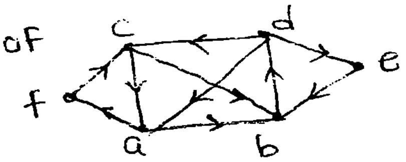
Traversability :-

A graph G is said to be traversal, if there exist a path which contain each edge of G exactly once and each vertex of G alleast once such a path is called Euler path.

Euler Circuit :- An Euler path in which the starting vertex is same as ending vertext of path is called Euler circuit.



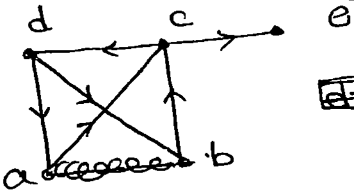
a-b-c-d-b-e-a-c-f-d-a



a-b-d-c-a-f-c-b-d-e-b-d-a

Theorm : A graph G is traversable, if no. of vertices with odd degree is exactly two or ~~more~~ zero.

- If graph G has exactly 2 vertices of odd degree then euler path exists but euler circuit does not exists.
- If graph G has no vertices of odd degree then Euler circuit also exists.
- If graph has exactly two vertices of odd degree then the euler path begins with one odd vertex and ends with other odd vertex.



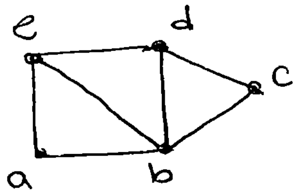
~~d-b-e~~

d-a-c-d-b-c-e

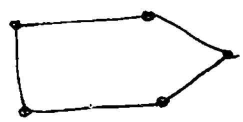
Hamiltonian graph :-

In a graph G, if there exists a cycle which contains each vertex of G exactly once, then the cycle is called Hamitonian cycle and the graph is called hamiltonian graph.

Ex:



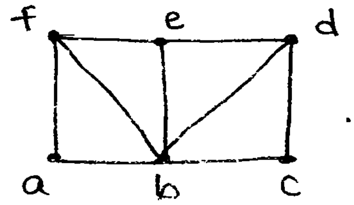
Ex:



- In Euler circuit every edge is present exactly once but in hamiltonian cycle some edges of the graph may be skipped.

* Hamiltonian graphs and traversable graph both are connected graph.

Ex : for the graph shown below which of the following is true?



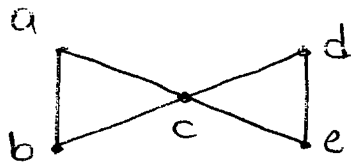
1. Euler path exists
2. Euler circuit exists
3. Hamiltonian cycle exists
4. Hamiltonian path exists

... of odd degree

... not traversible

Euler graph & circuit not exists.

Which is/are true?



In the graph no. of vertices with odd degree is zero

It is traversible and Euler circuit also exists.

Euler path (b) Euler circuit

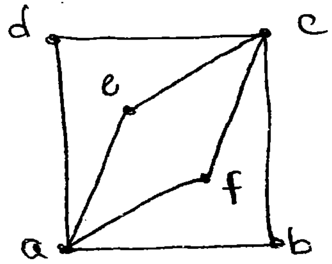
Ham cycle (d) Ham path

* In hamiltonian cycle degree of each vertex is 2.

Therefore to construct hamiltonian in G we have to delete 2 edges of vertex C, then we are left with 5 vertices & 4 edges ∴ a cycle graph with 5 vertices req. 5 edges.

Ham. path exists a-b-c-d-e which contains all 5 edges (Ham. circuit not exists)

Ex. which is/are true?



The graph has no vertices with odd degree

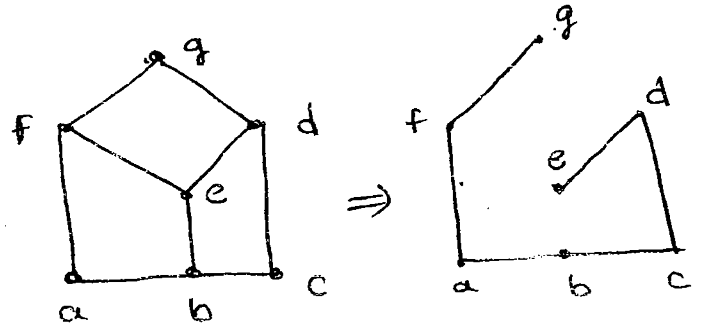
∴ It is traversable and Euler circuit also exists.

To construct ham. cycle for the graph we have to delete

two edges at each of the vertices a & c

Then we are left with 6 vertices and 4 - edges,
therefore neither ham. cycle nor path exists.

Ex:



Odd degrees of vertices present

∴ No Euler path and Circuit present.

To construct ham. cycle for the graph we have to delete one edge at each of the vertices b, d, f then left with 6 edges & 7 vertices

. ham.cycle is not possible. but ham. path exists.

Set theory & Algebra

* sets
* Relations
* functions
* Groups
* lattice
* Boolean Algebra

Set :- A well defined unordered collection of distinct elements.

$A = \{ x \mid x \text{ is an integer and } 1 \leq x \leq 10 \}$

$= \{ 1, 2, 3, \dots 10 \}$

Sets are denoted by upper case letters and elements denoted by lower case letters.

Null set :- A set with no elements.

It is denoted by ϕ.

Ex: $A = \{ x \mid x \text{ is a prime no. and } 8 < x < 10 \}$

$A = \{ \}$

$A = \phi$

Subset : If A and B are sets, such that every element of A is also an element of B then A is a subset of B.

Denoted by $A \subseteq B$

For any set A, $A \subseteq A$ and $\phi \subseteq A$

Proper subset :- Any subset of A, which is not a trivial subset of A is called proper subset of A.

Ex: $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$ then $A \subset B$

* If $A \subseteq B$ and $B \subseteq A$, then $A = B$

Power set :- Let A be a set then power set of A denoted by $P(A)$ = set of all subset of A.

Let $A = \{1, 2, 3\}$

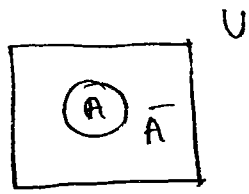
then $P(A) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\} \}$

Note : If $|A| = n$ then no. of elements in $P(A) = 2^n$

Universal set : set of all objects under discussion (U)

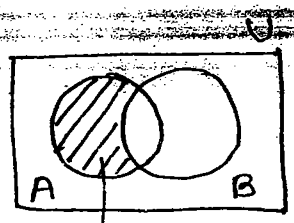
Complement of set :- A universal set U and a set A contained in U, then all elements that belong to U but not A is called complement of A (in U)

It is denoted by $\bar{A}$.



Set difference :- (Relative complements)

$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$



A-B

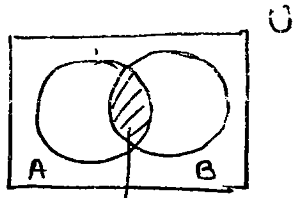
If $A = \{1, 2, 3, 4\}$

$B = \{2, 4, 5, 6\}$

$A - B = \{1, 3\}$

Set intersection :

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



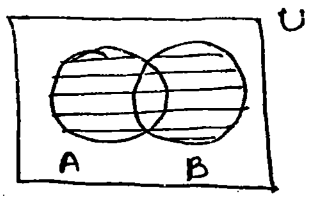
$A \cap B$

$A \cap B \subseteq A$ and

$A \cap B \subseteq B$

Set Union :

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



$A \subseteq (A \cup B)$ and

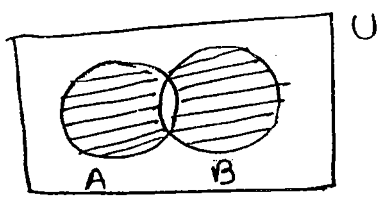
$B \subseteq (A \cup B)$

Symmetric Difference :- (Boolean sum)

$$A \,\Delta\, B = \{x \mid x \in A \text{ or } x \in B \text{ but } x \notin (A \cap B)\}$$

$A \,\Delta\, B = (A \cup B) - (A \cap B)$

$A \oplus B = (A - B) \cup (B - A)$



$A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3, 4, 5, 6\}$

$A \,\Delta\, B = \{5, 6\}$

The laws of set theory :

1. **commutative laws :**

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
$$A \oplus B = B \oplus A$$
$$A - B \neq B - A$$

2. **Associative laws :**

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$
$$A - B \neq B - A$$

3. **Distributive law :**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. **DeMorgan laws :-**

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$
$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$

5 **Idempotent Laws :-**

$$A \cup A = A$$
$$A \cap A = A$$

6. **Identities :-** $A \cup \phi = A$, $A \cup U = U$, $A \cup \overline{A} = U$

$A \cap U = A$ $\quad A \cap \overline{A} = \emptyset$, $A - B = A \cap \overline{B}$

7. **Absorption law :-** $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

8. **Modular laws :-**

$$A \cup (B \cap C) = (A \cup B) \cap C \quad \text{iff } A \subseteq C$$
$$\Rightarrow (A \cup B) \cap (A \cup C) = (A \cup B) \cap C$$
$$A \cup C = C$$
$$A \subseteq C$$
$$A \cap (B \cup C) = (A \cap B) \cup C \quad \text{iff } C \subseteq A$$

Note : * $P(A \cap B) = P(A) \cap P(B)$

* $P(A \cup B) \neq P(A) \cup P(B)$

* $A - (A - B) = A \cap B$

* $A - (A \cap B) = A - B$

Ex: which of the following is not true ?

a) IF $A \subseteq \phi$, then $A = \phi$ —— true

b) $B \cup (A \cap B) = B$ —— true

$A \cap B \subseteq B$

c) $(A \cap B) \cup (A \cap B^C) = A$

$L \cdot H \cdot S = A \cap (B \cup B^C)$

$= A \cap U$

$= A = R \cdot H \cdot S$ —— True

$(A \cdot B) + (A \cdot \overline{B})$

$= A \cdot (B + \overline{B}) = A$

d) $(A \cap B) \cup (A \cap B^C) \cup (A^C \cap B) \cup (A^C \cap B^C) = A \cup B$

$(A \cdot B) + (A \cdot \overline{B}) + (\overline{A} \cdot B) + (\overline{A} \cdot \overline{B})$

$= A \cdot (B + \overline{B}) + \overline{A} \cdot (B + \overline{B})$

$= A + \overline{A}$

$= 1$

$= U \neq A \cup B$ —— False.

Ex: IF $A \subset B$ then, which of the following is not true ?

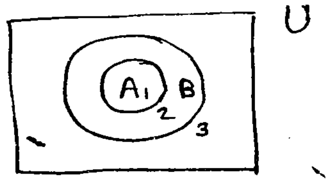
a) $B^C \subset A^C$

b) $A \cup B = B$

c) $A \cap B = A$

~~d) $A \cap B = A$~~

d) $B - A = \phi$



$A \subset B$

Ex: IF $A \oplus B = (A - B) \cup (B - A)$ which is False ?

a) $A \oplus A = \phi$

b) $A \oplus \phi = A$

c) $(A \oplus B) \oplus B = A$

$L \cdot H \cdot S. = A \oplus (B \oplus B)$

$= A \oplus \phi$

$= A = R \cdot H \cdot S.$

d) $A \oplus B = A$ iff $B = \phi$

$\Rightarrow A \oplus B = A$

$A \oplus B = A \oplus \phi$

e) $A \oplus B = (A \cap B^C) \cap (B \cap A^C)$

Ex: Let ϕ be the empty set, then $|P\{P(\phi)\}| = ?$

$$\phi = \{\ \}$$

$$P(\phi) = \{\phi\}$$

$$P\{P(\phi)\} = \{\phi, \{\phi\}\}$$

$$|P\{P(\phi)\}| = 2$$

Ex: Consider $A = \{ S \mid S \text{ is a set} \}$

$B = \{ S \mid S \text{ is a set and } S \notin S \}$

which is true ?

a) A is a set and B is not a set

b) B is a set and A —— " ——

c) Both A and B are sets

d) Both A and B are not sets.

$\Rightarrow$ * $\boxed{\text{Set of set is not set.}}$ (An set can not be element of itself)

Ex: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$ are represented by the bit strings 1111100000 and 1010101010 resp. $(U = \{1, 2, 3, \ldots 10\})$

$= \{1111111111\}$ (missing bit 0)

a) $A \cup B = \underset{1\,2\,3\,4\,5\,6\,7\,8\,9\,10}{1111101010}$

$A \cup B = \{1, 3, 5, 2, 4, 7, 9\}$

b) $A \cap B = \underset{1\ \ 3\ \ 5}{1010100000}$

$A \cap B = \{1, 3, 5\}$

c) $A - B = \underset{2\ \ 4}{0101000000}$

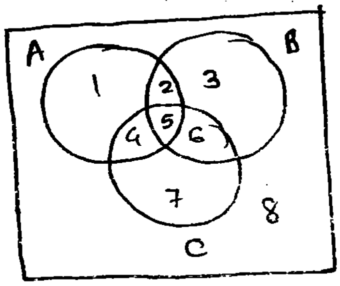
$A - B = \{2, 4\}$

d) $B^c = 0000011111$

$B^c = (2, 4, 6, 8, 10)$

$= 0101010101$

Ex: Which is False?



a) $(A-B)-C = (A-C)-(B-C)$

$\{1,4\} - \{4,5,6,7\} = \{1\}$

$\{1,2\} - \{2,3\} = \{1\}$

True

b) $(A-B)-C = (A-C)-B$

$\{1,4\} - \{4,5,6,7\} = \{1\}$ | $\{1,2\} - \{2,3,5,6\} = \{1\}$

TRUE

c) $(A\cap B)-(B\cap C) = (A-(A\cap C))-(A-B)$

$\{2,5\} - \{5,6\} = \{2\}$ | $[\{1,2,4,5\} - \{4,5\}] - \{1,4\} = \{1,2\} - \{1,4\} = \{2\}$

TRUE

d) $(A-B)-C = A-(B\cap C)$

$\{1,4\} - \{4,5,6,7\} = \{1\}$ | $\{1,2,4,5\} - \{5,6\} = \{1,2,4\}$

$\neq$

false

Multiset :- A unordered collection of object in which any object can appears more than once is called Multiset.

E.g. $\{a, a, b, b, b, c, c, c, c, d\}$

$= \{2a, 3b, 4c, d\}$

Let, $P = \{m_1 a_1, m_2 a_2 \ldots m_k a_k\}$

where m_i = multiplicity of a_i

$Q = \{n_1 a_1, n_2 a_2 \ldots n_k a_k\}$

$P \cup Q$ = A multiset in which the multiplicity of $a_i = \max\{m_i, n_i\}$

$P \cap Q$ = —— " —— of $a_i = \min\{m_i, n_i\}$

$P - Q$ = A multiset of $a_i = \begin{cases} m_i - n_i & ; \text{if } m_i > n_i \\ 0 & \text{otherwise} \end{cases}$

$P + Q$ = A multiset of $a_i = m_i + n_i$

Ex: $A = \{3a, 2b, 1c\}$, $B = \{2a, 3b, 4d\}$

$A \cup B = \{3a, 3b, 1c, 4d\}$

$A \cap B = \{2a, 2b\}$

$A - B = \{a, c\}$

$A + B = \{5a, 5b, 1c, 4d\}$

Ex: The set $A = \{1, 2, 3 \ldots n\}$ then how many multisets are possible with the element A.

a) 2^n b) n^n c) $2^{(n^2)}$ d) unlimited

→ Each no. can appears many no. of times

∴ Multisets are unlimited.

Ex: If $A = \{1, 2, \ldots n\}$, then how many multisets of size 4 are possible with elements of A, so that atleast one element appear exactly two.

≥ 1

→ $\{a, b, c, d\}$ — $\{a, a, b, c\}$

$\{a, a, b, b\}$

Case 1 : ~~only one elements appears exactly twice.~~

In this case the multiset is ~~of the~~ form $\{a, a, b, c\}$

No. of multiset in this case $= {}^{n}C_3 \cdot {}^{3}C_1$ or ${}^{n}C_1 \; {}^{(n-1)}C_2$

case 2 : Two Elements appears exactly twice

No. of Multisets in this case $= {}^{n}C_2$

Req. No. of multisets $= {}^{n}C_3 \cdot {}^{3}C_1 + {}^{n}C_2$

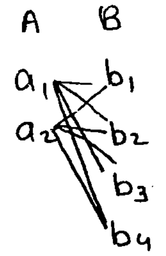
$$= \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \times 3 + \frac{n(n-1)}{2}$$

$$= \frac{n(n-1)^2}{2}$$

Cartesian Product :-

- If A and B are two sets, then the cartesian product of A and B defined as $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- If no. of elements in A is m and no. of elements in B is n then no. of elements in $A \times B$

  $|A \times B| = mn$



- In General $(A \times B) \neq (B \times A)$
- If $(A \times B) = (B \times A)$ then $A = B$ or $A = \emptyset$ or $B = \emptyset$

Relation :-

- A and B are two sets then Every subset of $(A \times B)$ is called a relation from A to B

$A \times B = \{(a_1, b_1)(a_1, b_2)(a_1, b_3)$
$(a_2 b_1)(a_2 b_2)(a_2 b_3)\}$

A →	B
a_1	b_1
a_2	b_2
	b_3

* If $|A| = m$ and $|B| = n$, then number of relations possible from A to B $= 2^{mn}$

A →	A
a_1	a_1
a_2	a_2
a_3	a_3

* A relation on set A to A is called a relation on A
* If $|A| = n$ then no. relations possible on $A = 2^{n^2}$

Inverse of a relation :-

If R is a relation from A to B, then

$R^{-1} = \{$ set of all order pairs $(b,a) \mid \cancel{(a,b)}\ (a,b) \in R \}$ is

a relation from B to A.

Complement of a relation :-

If R is a relation from A to B, then

$\bar{R} = (A \times B) - R$

Diagonal Relation :-

A relation R on set A is said to be diagonal relation, if $R = \{(x,x) \mid x \in A\}$

$= \Delta A$

$A \longrightarrow A$

$a_1 \longrightarrow a_1$

$a_2 \longrightarrow a_2$

$a_3 \longrightarrow a_3$

$R = \{(a_1, a_1)\ (a_2 a_2)\ (a_3 a_3)\}$

Reflexive Relation :-

A relation R on set A is said to be reflexive if

$x R x \quad \forall x \in A$

i.e. $(x,x) \in R \quad \forall x \in A$

* A diagonal relⁿ on set A is reflexive and every superset of diagonal relⁿ is also reflexive.
* Reflexive relⁿ may contains non-diagonal relⁿ.

Ex: $A = \{1,2,3\}$ then $R_1 = \{(1,1)(2,2)(3,3)\}$ — smallest reflex relⁿ (Δ_A)

$R_2 = \{(1,1), (2,2)(3,3)(1,2)(3,1)\}$

$R_3 = A \times A$ — Large reflex relⁿ

Ex: If $A = \{1 \ldots n\}$ then no. of Reflexive relⁿ possible on $A = ?$

→ No. of non diagonal pairs $= n^2 - n = n(n-1)$

No. of refexive relⁿ possible $= 2^{n(n-1)}$

- The relation $\leq$ is reflexive on any set of real no.s.
- The relation $\subseteq$ is reflexive on any collection of sets.
- The relation 'divides' or 'is a divisor of' denoted by '|' is reflexive on any set of non zero real no.s.

IRREFLEXIVE RELATION :- A reln R on a set A is said to be irreflexive if $x \not R x \quad \forall x \in A$

i.e. $(x, x) \notin R \quad \forall x \in A$

Ex: $A = \{1, 2, 3\}$ then $R_1 = \{\phi, (1,2), (2,3), (3,2)\}$

* Smallest irreflexive reln on A = ϕ relation.
* Largest —— " —— $= (A \times A) - \Delta_A$
* if $A = \{1 - - n\}$ then no of irreflexive reln $= 2^{n(n-1)}$
* The relations "is less than" and "is greater than" are irreflexive on set of all real no.s.
* $R = \{(1,1), (2,3)\}$ on $A = \{1, 2, 3\}$ is ~~nor~~ neither reflexive nor irreflexive.

Symmetric Relation :- A reln R on a set A is said to be symmetric if (xRy) and (yRx) $\forall x, y \in A$. That is if $(x, y) \in R$ then $(y, x) \in R \quad \forall x, y \in A$. ↳ y related to x

- Ex: if $A = \{1, 2, 3\}$ then $R = \{(1,1), (3,3), (1,2), (2,1)\}$ is a symmetric reln on A.

  $R = \{(1,2), (2,1), (1,3)\}$ is not symmetric because $(1,3) \in$ but $(3,1) \notin R$.

* smallest reln on A = Empty reln (ϕ)
* Largest —— " —— $= A \times A$
* if $A = \{1 - - n\}$ then no. of symmetric relations are possible on $A = 2^n \cdot 2^{[n(n-1)]/2}$

  $= 2^{[n(n+1)]/2}$

* The relation x is "a complement of y" is symmetric in Boolean algebra.
* The relation x is "perpendicular to" is symmetric for any two lines in a plane.
* The relation x is "brother of" is symmetric for any

Antisymmetric Relation :- A relation R on a set A is said to be antisymmetric if $(a,b) \in R$ and $(b,a) \in R$ then $a=b$

* if R is not antisymmetric if there exists $a,b \in A$ such that $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$.
* In antisymmetric relⁿ only one of the pair of (x,y) or (y,x) can be present or both of the pairs may be ~~present~~ absent.

  Eg: if $A = \{1,2,3\}$ then $R = \{(1,1), (2,2), (2,1), (2,3)\}$ is a Antisymmetric relation.

* smallest Antisymmetric relⁿ on A = Empty set ϕ.
* No. of elements in a largest antisymmetric relation on

  $A = \dfrac{n(n+1)}{2}$ $(n(n+1)/2)$

* The relⁿ is $a \leq b$ and $b \leq a$ then $a=b$ is antisymmetric relⁿ on the set of all real no.
* if $A = \{1 \cdots n\}$ the no. of antisymmetric relⁿ possible on A

  $= \underline{\underline{2^n \cdot 3^{[n(n-1)]/2}}}$

* $R = \{(1,1), (3,3)\}$ — symmetric also ~~also~~ antisymmetric.
* $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (3,1)\}$ — largest antisymm.

$\longrightarrow \{x,y\}$ → (x,y), (y,x), — $\Big\}$ 3 possibilities with non diagonal pairs

Ex: If $A = \{1,2,3\}$ then how many relⁿ on A are not antisymmetric.

$\longrightarrow$
$$= 2^{(n^2)} - 2^n \cdot 3^{n(n-1)/2} \quad \text{where } n=3$$
$$= 2^9 - 2^3 \cdot 3^3$$
$$= 512 - 216$$
$$= 296$$

* The relⁿ $\leq$ is antisymmetric on any set of real no.s.
* The relⁿ $\subseteq$ is antisymmetric on any collection of sets.
* The relⁿ $|$ is antisymmetric on any set of +ve integers.

Asymmetric Relation :- A relation R on a set A, is said to be asymmetric if (xRy) then $(y \not R x)$ $\forall x, y \in A$

i.e. if $(x,y) \in R$ then $(y,x) \notin R$ $\forall x, y, \in A$

* Every Asymmetric relation is also antisymmetric.
* In asymmetric relation diagonal pairs are not allowed where as in antisymmetric relation diagonal pairs can be present.

Ex: $A = \{1,2,3\}$ then $R_1 = \{\ \}$ — smallest

$R_2 = \{(1,2), (2,3)\}$

$R_3 = \{(1,2), (3,2), (1,3)\}$

* If $A = \{1 \cdots n\}$ then no. of asymmetric relations possible on $A = 3^{n(n-1)/2}$
* The relation $<$ is asymmetric on any set of real no-s.
* ——"—— $>$ ——"—— "——"——.

Transitive Relation :- A relation R on a set A is said to be transitive if $(xRy$ and $yRz)$ then (xRz) $\forall x, y, z \in A$.

$\downarrow$

x related to z

* The relation $\leq$ is transitive on any set of real no-s.
* ——"—— $|$ is ——"—— "——.
* ——"—— $\subseteq$ is ——"—— collection of sets.

Ex: If $A = \{1,2,3\}$ then $R_1 = \{\ \}$ — smallest

$R_2 = \{(1,1), (2,2)\}$

$R_3 = \{(1,2), (2,3), (1,3)\}$

$R_4 = A \times A$ — largest.

$R_5 = \{(1,2), (2,1), (1,1), (2,2)\}$

Ex: If $A = \{1,2\}$ then no. of transitive relⁿ possibles on A = ?

→ Relⁿ which are not transitive are $R_1 = \{(1,2), (2,1)\}$

$R_2 = \{(1,2), (2,1), (1,1)\}$

$R_3 = \{(1,2), (2,1), (2,2)\}$

$\therefore$ Transitive reln $= 2^{n^2} - 3$

$= 2^4 - 3 = 16 - 3$

$= 13$

Equivalence Relation :- A reln R on a set A is said to be an equivalence reln if R is refexive, transitive and symmetric.

Ex : If $A = \{1, 2, 3\}$ then how many equivalence reln are possible on A = ?

$\rightarrow$

$R_1 = \{(1,1)(2,2)(3,3)\}$ — smallest equivalence reln on A

$R_2 = \{(1,1)(2,2)(3,3), (1,2)(2,1)\}$

$R_3 = \{(1,1),(2,2),(3,3),(2,3),(3,2),\}$

$R_4 = \{(1,1),(2,2),(3,3), (1,3),(3,1)\}$

$R_5 = A \times A$ — largest equivalence reln of A.

$\therefore$ 5 equivalence reln are possible.

Ex: If $A = \{1, 2, 3, 4\}$ then no. of equivalence reln possible on A = ?

$\rightarrow$ 15.

Ex: A reln R on set of all real no. is defined by

$x R y \Leftrightarrow (x-y)$ is integer then R is an equivalence relation.

$\rightarrow$ 1) we have $x - x = 0$, an integer

$\Rightarrow x R x \quad \forall x$

$\therefore$ R is reflexive.

2) Let $x R y$

$\Rightarrow (x-y)$ is an integer

$\Rightarrow (y-x)$ —— " ——

$\Rightarrow y R x$

$\Rightarrow$ R is symmetric

3) Let $x R y$ and $y R z$

$\Rightarrow (x-y)$ & $(y-z)$ is an integer

Now, $(x-z) = (x-y) + (y-z)$

$\Rightarrow (x-z)$ is an integer

Ex: A relⁿ R on set of all integers is defined by

$x R y \iff (x-y)$ is an even no. then R is an equivalence relⁿ.

Ex: A relⁿ R on set of all integers is defined by

$x R y \iff (x-y)$ is divisible by 5 then R is an equivalence relⁿ.

Partial ordering relation :-

A relation R on a set A is said to be partial ordering relation if R is reflexive, antisymmetric, and transitive.

Partially ordered set (Poset) :- A set 'A' with partial ordering relation R defined on A, is called Poset and denoted by [A; R]

~~Ex: If A~~

* The relation $\le$ is a partial ordering relation, on any set of real no.s A. i.e. $[A; \le]$ is a poset.
* The relation $|$ is a partial ordering relⁿ on any set of +ve integer no.s. i.e. $[A; |]$ is a poset.
* The relation $\subseteq$ is a partial ordering relⁿ on any collection of sets 'S' i.e. $[S; \subseteq]$ is a poset.

Totally ordered set :- A poset [A; R] is called totally ordered set, if every pair of elements are compared w.r.t. R.

- Also called as linearly ordered set or chain.

i.e. aRb or bRa $\forall a, b \in A$

Ex: If A is any set of real no. then the poset $[A; \le]$ is a totally ordered set.

Ex: Let $A = \{1, 2, 3 ---- 10\}$ then poset $[A; |]$ is not a totally ordered set.

Ex: —"— $A = \{1, 4, 8, 16, 32\}$ ——"—— ——"—— is a totally ordered set.

Ex: If $A = \{1,2\}$ then the poset $[P(A) ; \subseteq]$ is not a totally orderd set

$\rightarrow$ $P(A) = \{ \phi, \underbrace{\{1\}, \{2\}}_{\text{not comparable}}, \{1,2\} \}$

Ex: If $A = \{ \phi, \{1\}, \{1,2\}, \{1,2,4\} \}$ the poset $[S ; \subseteq]$ is a totally ordered set.

Ex: Let $A = \{1,2,3\}$ R_1 R_2 R_3 R_4 are relations on A.
which of the following is false ?

a) $R_1 = \{ (1,1), (3,3) \}$ is symmetric and antisymmetric.
b) $R_2 = \{ (1,2), (2,1), (3,2) \}$ is neither symmetric nor antisymmetric
c) $R_3 = \{ (1,2), (2,3), (3,2) \}$ is not symmetric but antisymmetric.
d) $R_4 = \{ (1,2), (2,1), (3,3) \}$ is symmetric but not antisymmetric.

$\therefore$ All stat. are true.

Ex: A rel^n R on a set $A = \{1,2,3,4\}$ is given by

$R = \{ (1,1), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4) \}$

is ___

(a) Equivalence Relation
(b) irreflexive, symmetric, transitive
(c) irreflexive, symmetric, antisymmetric
(d) Transitive

irreflexive: not becoz (2,2) diagonal present

symmetric: Not For (2,1) (1,2) not present

antisymmetric: not becoze (2,3) (3,2)

Ex: Relation R on set of all integers is defined by $a^R b \Leftrightarrow b = a^k$ for some +ve integer K. then which is true ?

(a) R is an equivalence relation
(b) R is a partial ordering rel^n
(c) R is total ordering rel^n
(d) R is a compatibility rel^n
$\hookrightarrow$ reflexive & symmetric but not transitive

→ i) we have, $a = a^1$

$\Rightarrow a R a \quad \forall a$

$\Rightarrow$ R is reflexive

Rel^n is not symmetric on set of all integers

Ex: $8 R 2$ but $2 R 8$ not

$\Downarrow$

$2^3 = 8$ but $2 = 8^?$

2) Let aRb and bRa

$\Rightarrow b = a^{k_1}$ and $a = b^{k_2}$ —— ①

$\Rightarrow \quad a = (a^{k_1})^{k_2}$

$a = a^{k_1 k_2}$

$\Rightarrow k_1 k_2 = 1$

$\Rightarrow k_1 = 1$ & $k_2 = 1$

$\therefore$ $b = a$ and $a = b$ (from ①)

$\therefore$ R is antisymmetric.

3) $b = a^{k_1}$ and $c = b^{k_2}$ —— ①

$\Rightarrow c = a^{k_1 k_2}$

$\Rightarrow aRc$

$\therefore$ R is transitive.

The given rel^n is not a total order.

Ex: $2 \not{R} 3$ & $3 \not{R} 2$

Ex: If $A = \{a, b, c\}$ then $R_1 = \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

is compatibility relation.

Ex: If A = {cat, bat, book, dog} then rel^n R on A is defined by word$_1$ R word$_2$ $\Leftrightarrow$ The two words have a common letter

then R is compatibility rel^n on ~~R~~ A.

→

cat R bat and bat R book

but cat $\not{R}$ book (no common letters)

Ex: which of the following statements is not true?

a) If a relⁿ R is symmetric and transitive then it is Reflexive — FALSE

→ If $A = \{1, 2, 3\}$ then

$R = \{(1,1), (2,2), (1,2), (2,1)\}$

symmetric, transitive but not reflexive.

b) if R is relⁿ on A is irreflexive and transitive then R is antisymmetric. — TRUE

$A = \{1, 2, 3\}$

$R = \{\ \}$

→ Let R is irreflexive and transitive but not antisym.

Let $(x,y) \in R$ and $(y,x) \in R$

(if R is not antisym. then we get two pairs (x,y) and $(y,x) \in R$).

$\Rightarrow$ $(x,x) \in R$ — by transitivity

$\Rightarrow$ R is not irreflexive.

which is contradiction.

$\therefore$ given stat. is true.

c) If R is antisymmetric relation on A, then $(R \cap S)$ is also antisymmetric for any relⁿ S on A. — TRUE

→ $R = \{(1,1), (2,3), (3,1)\}$

Every subset of antisym. is antisym.

$(R \cap S) \subseteq R$

$\therefore$ $R \cap S$ is also antisym.

d) If R is reflexive then $\overline{R}$ is irreflexive. — TRUE

→ $\overline{R} = (A \times A) - R$

Ex: which of the following stat. is not true?

a) If R and S are reflexive relⁿ on A, then (R∩S) and (R∪S) are reflexive → <u>TRUE</u>

→ for (R∩S) & (R∪S) — diagonal pairs are present.

b) If R and S are ~~anti~~ symmetric then (R∪S) & (R∩S) are also symmetric. → <u>TRUE</u>

→ $R = \{(a,b), (b,a)\}$

$S = \{(a,b), (b,a), (a,c), (c,a)\}$

$R \cap S = R$

$R \cup S = \{\text{all}\}$

∴ symmetric.

c) If R and S are transitive then (R∪S) & (R∩S) are transitive.
↳ <u>FALSE</u>

→ $R = \{(a,b)\}$ → always transitive

$S = \{(b,c)\}$

$R \cup S = \{(a,b), (b,c)\}$

↓

(a,c) absent

∴ R∪S is not transitive

$R \cap S = \{\}$ → transitive

d) If R and S are antisym. relⁿ on A, then (R∪S) need not be antisym. but (R∩S) is always antisym.

→ $R = \{(a,b)\}$

$S = \{(b,a)\}$

$R \cup S = \{(a,b), (b,a)\}$ — Not antisym.

$R \cap S = \{\}$ — antisym.

Transitive closure :- Let R be any relⁿ on set A

Transitive closure of R is denoted by R^* is defined by = the smallest transitive relⁿ on A which contains R.

Ex: Let $A = \{1,2,3\}$ and $R = \{(1,2),(2,3)\}$

then $R^* = \{(1,2),(2,3),(1,3)\}$

if R is transitive relⁿ then $R^* = R$

Reflexive closure :-

Reflexive closure of $R = R^{\#}$
= The smallest reflexive relation on A which contain R
$= R \cup \Delta_A$

Ex: $A = \{1,2,3\}$
$R = \{(1,2),(2,3)\}$
then $R^{\#} = \{(1,2),(2,3),(1,1),(2,2),(3,3)\}$

Symmetric closure :-

Symmetric closure of $R = R^{\cancel{\oplus}+}$
= The smallest symmetric relⁿ on A which contain R.
$= R \cup R^{-1}$

Ex: $A = \{1,2,3\}$
$R = \{(1,2),(2,3)\}$
then $R^+ = \{(1,2),(2,3),(2,1),(3,2)\}$

Note :-

1) **Symmetric closure of reflexive ~~clouse~~ closure of R :-**

= Reflexive closure of symmetric closure of R.

Ex: $A = \{1,2,3\}$
$R = \{(1,2),(1,1),(2,2),(3,3)\}$ — Reflexive
[illegible] $(2,1)\}$ — Symmetric

② Transitive closure of the reflexive closure of R
= Reflexive closure of the transitive closure of R.

Ex: $A = \{1,2,3\}$

$R = \{(1,2), (2,3)\}$

$R_1 = \{(1,2), (2,3), (1,3)\}$ — transitive

$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,3)\ (1,3)\}$

— Reflexive clos. of trans. closure.

③ Transitive closure of the symmetric closure of R need not be symmetric closure of the transitive colsure of R.

Let $A = \{1,2,3\}$

$R = \{(1,2)\ (2,3)\}$

$L.H.S. = \{(1,2)\ (2,3)\ (2,1)\ (3,2)\ (1,3)\ (3,1)$
$(1,1)\ (2,2)\ (3,3)\}$

$= A \times A$

$R.H.S = \{(1,2)\ (2,3)\ (1,3)\}$ — transitive

$= \{(1,2)\ (2,3)\ (1,3)\ (2,1)\ (3,2)(3,2)\}$

└ symm. clos. of transitive clo-

$\neq A \times A$

$LHS \neq RHS$

Ex: $R = \{(x,y) \mid y = x+1$ and $(x,y) \in \{0,1,2,---\infty\}\}$
then the reflexive transitive closure of R = ?

→ $R = \{(0,1), (1,2), (2,3), (3,4) ---\}$
$(0,2), (0,3), (0,4)$ - - - -
$(1,3)\ (1,4)\ (1,5)$ - -
$(2,4)\ (2,5)\ (2,6)$ - - -
$(0,0)\ (1,1)\ (2,2)\ (3,3)$ - - - -

∴ $\{(x,y) \mid y \geq x$ and $(x,y) \in (0,1,2,---)\}$

Ex: Let $A = \{1,2,3\}$ and a relation R on A is defined by $R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$ then transitive closure of R is —

→ The matrix corresponding to given relⁿ is

Warshal's Algorithm

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \xRightarrow{+(3,3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & \oplus 1 \end{bmatrix}$$

	I	II	III
Column	{1,3}	{2,3}	{1,3}
Row	{1,3}	{2}	{1,2,3}

(1,1)   {2,2}   {1,1} {1,2} {1,3}
(1,3)   {3,2}   {3,1} {3,2} {3,3}
(3,1)
(3,3)

$\therefore \quad R^* = \{(1,1), (1,3), (2,2), (3,1), (3,2)\ (3,3)\ (1,2)\}$

$= (A \times A) - \{(2,1), (2,3)\}$

Ex: Let $A = \{a,b,c,d\}$ and relⁿ R on the set A is defined by $R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$ Find transitive closure of R.

→ Matrix $=$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & & & \\ d & & & & \end{array}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xRightarrow[\substack{+\{d,a\} \\ +\{d,d\}}]{+\{b,d\}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \cancel{0}\,1 & 0 & 1 & \cancel{0}\,1 \end{bmatrix}$$

	I	III	II	IV	
column	{b,c}	{b,d}	{ }	~~{a,c}~~	{a,b,c,d}
Row	{d}	{a,d}	{a,c}	~~{c}~~	{a,c,d}

{b,d}   {b,a} {d,a}
{c,d}   (b,d) (d,d)

{a,c} {a,a} {a,c} {a,d}
{c,c} {b,a} {b,c} {b,d}
{c,a} {c,d} {d,a}
{d,c} {d,d}

$R^* = \{(a,d), (b,a), (b,c), (c,a),$
$(c,d), (d,c), (b,d), (a,a),$
$(a,c)\ (d,a)\ (d,d), (c,c)\}$

Equivalence classes :- Let R be an equivalence Relⁿ for a set A, for any element $x \in A$, equivalence class of A denoted by $[x]$, is defined as

$$[x] = \{ y \mid y \in A \text{ and } x R y \}$$

Note :* We can have $[x] = [y]$ even though $x \neq y$

* Set of all distinct equivalence classes of A define a partition of A.

Partition of a set :- Let A be a set with n-elements $(n \geq 2)$. A subdivision of A, $\{A_1, A_2, --- A_k\}$ into nonempty and non overlapping subset is called a partition if $A_1 \cup A_2 \cup A_3 \cup ---- \cup A_k = A$

Ex: $A = \{1,2,3,4,5\}$

$P_1 = \{(1,2), (3,4), (5)\}$ (No common element is present).

Ex: An equivalence relation on the set $A = \{a,b,c,d,e,f\}$ is defined by $R = \{(a,a), (b,b), (c,c), (d,d), (e,e), \cancel{(a,d)}, (f,f)$ $(a,f) (f,a) (b,e)$ $\cancel{(d,a)}, \cancel{(b,a)}, \cancel{(a,b)\}}$ $\cancel{(e,d)}, \cancel{(d,e)}$ $(e,b)\}$

Find partition of A defined by R.

$\rightarrow$

$[a] = \cancel{\{a,b,d\}}$ $\{a,f\}$

$[b] = \cancel{\{b,a,}$ $\{b,e\}$

$[c] = \{c\}$

$[d] = \{d\}$

$[e] = \cancel{\{e\}}$ $\{e,b\}$

$[f] = \{a,f\}$

distinct

Required partition $= \{(a,f), (b,e), (c), (d)\}$

Ex: Let $A = \{1,2,3,4,5\}$ and a partition of A is given as $P = \{(1,3), (2,5), (4)\}$ Find equivalence relⁿ of A w.r.to. P.

$\rightarrow$ $R = \{(1,3) \times (1,3), (2,5) \times (2,5), (4) \times (4)\}$

$R = \{(1,1), (1,3), (3,1), (3,3), (2,2), (2,5), (5,2), (5,5), (4,4)\}$

Ex : Let A = set of real no.s, An equivalence relⁿ R on A is defined by $x R y \Leftrightarrow (x-y)$ is an even no. then $[1] = ?$

→ $[x] = \{ y \mid (x-y)$ is an even no.$\}$

$[1] = \{ y \mid (1-y)$ is —— " ——$\}$

$= \{ \pm 1, \pm 3, \pm 5, ---- \}$

= set of all odd numbers.

$[2] = \{ y \mid (2-y)$ is —— " ——$\}$

$= \{0, \pm 2, \pm 4, \pm 6 ---- \}$

= set of even no.s.

Ex : Let A = set of real no.s, An equivalence relⁿ R on A is defined by $x R y \Leftrightarrow (x-y)$ is integer then

i) what is the equivalence class of 1 ?

ii) $[1/2] = ?$

→ $[x] = \{ y \mid x-y$ is integer$\}$

$[1] = \{ y \mid (1-y)$ is integer$\}$

= set of all integers

$[1/2] = \{ y \mid (1/2 - y)$ is integer$\}$

$= \{ \pm 1/2, \pm 3/2, \pm 5/2 --- \}$

= odd multiple of 1/2

$= \{ (n + 1/2) \mid n$ is any integer$\}$

Ex : Let A = set of all integers

$x R y \Leftrightarrow (x-y)$ is divisible by 3. How many distinct equivalence classes are possible on A w.r.to R ?

→ $[x] = \{ y \mid (x-y)$ is divisible by 3$\}$

$[0] = \{ --- -9, -6, -3, 0, 3, 6, 9 --- \}$

$[1] = \{ ---- -8, -5, -2, 1, 4, 7, 10 --- \}$

$[2] = \{ --- -7, -4, -1, 2, 5, 8, 11, --- \}$

$A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\}$

Lattice Properties :- A lattice is denoted by $[L, \cup, \cap]$

$\forall\ a, b, c \in L$

1) $a \cup b = b \cup a$
   $a \cap b = b \cap a$

2) $(a \cup b) \cup c = a \cup (b \cup c)$
   $(a \cap b) \cap c = a \cap (b \cap c)$

3) $a \cup a = a$
   $a \cap a = a$

4) $a \cup (a \cap b) = a$
   $a \cap (a \cup b) = a$
   $b \cup (b \cap a) = b$
   $b \cap (b \cup a) = b$

5) In a lattice L, $a \cup b = b \Leftrightarrow (a \cap b) = a \quad \forall\ a, b \in L$

Sub lattice : Let $[L, \cup, \cap]$ be a lattice, a subset M of L is called a sublattice of L if

i) $[M, \cup, \cap]$ is a lattice

ii) LUB of (GLB) of any two elements a and b of M is same as LUB of a and b in L.

Distributive lattice : A lattice L is said to be distributive if

$$\left.\begin{aligned} a \cup (b \cap c) &= (a \cup b) \cap (a \cup c) \\ a \cap (b \cup c) &= (a \cap b) \cup (a \cap c) \end{aligned}\right\} \forall\ a, b, c \in L$$

Bounded lattice : Let M be lattice w.r.t. Relⁿ R

if there exist an element $I \in L$ such that

$a R I \quad \forall\ a \in L$ then I is called upper bound of L

Similarly, if there exist element $O \in L$ such that

$O R a \quad \forall\ a \in L$ then O is called lower bound of L

In a lattice L if upper bound and lower bound exists then it is called Bounded Lattice.

– In a bounded lattice the upper bound (and lower bound) is unique.

In a bounded lattice L, following properties hold good :-

$$\left.\begin{aligned} a \cup I &= I \\ a \cap I &= a \end{aligned}\right\} \forall a \in L$$

$$\left.\begin{aligned} a \cup 0 &= a \\ a \cap 0 &= 0 \end{aligned}\right\} \forall a \in L$$

Complement :- Let L be a bounded lattice for element $a \in L$ if there exists an element $b \in L$

such that $\left.\begin{aligned} a \cup b &= I \\ a \cap b &= 0 \end{aligned}\right\}$ then b is called complement of a

* In a lattice complement of an element may or may not exists and need not be unique if exists.

Complemented Lattice :- In a lattice if each element has a complement, then it is called complemented lattice.

* In a complemented lattice each element has atleast one complement.

* In a distributive lattice complement of element if exists, then it is unique.

* In a distributive lattice each element has atmost one complement.

Boolean Algebra :- If a lattice is distributive and complemented then it is called Boolean algebra / Boolean lattice.

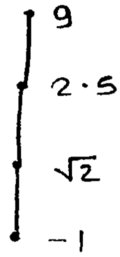
* In a Boolean Algebra each element has unique complement.

Hasse Diagram (Poset Diagram) :-

Let [A ; R] be a finite poset on the hasse diagram of A,

if
1) There is a vertex corresponding to each element of A
2) An edge betn two elements a and b is not present iff there is an element x such that $a R x$ and $x R b$
3) An edge betn a and b is present iff $a R b$ and there is no element x such that $a R x$ and $x R b$.

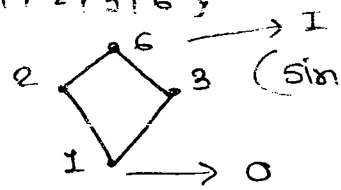
Ex: Draw Hasse dia. for the poset $[A; \leq]$ where $A = \{-1, \sqrt{2}, 2.5, 9\}$



* In a total divided set complement exists only for lower bound and upper bound (Not a complemented lattice).

Ex: Draw Hass dia. for the poset $[D_6 ; |]$ where

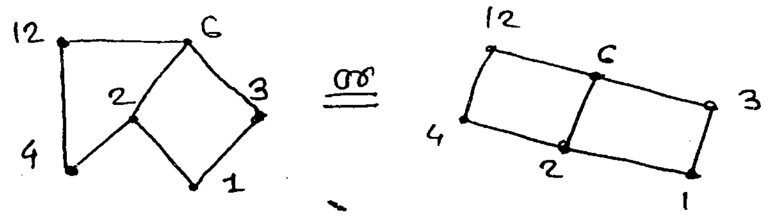
$D_6 = \{1, 2, 3, 6\}$



(Since (2 & 3) present direct edge 1 to 6 not allowed)

Ex: Draw Hass. Dia. for $[D_{12} ; |]$

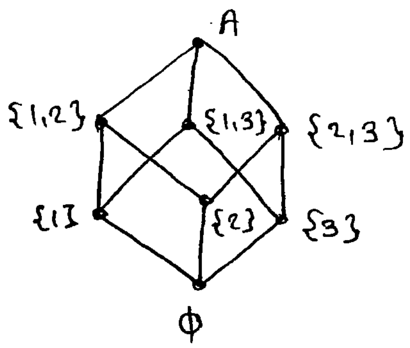
$D_{12} = \{1, 2, 4, 3, 6, 12\}$



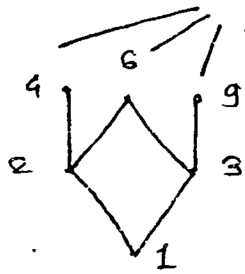
Ex: Draw Hass. Dia. $[P(A) ; \subseteq]$

where $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\} \{1,3\} \{2,3\}, A\}$



Ex: Draw the Hasse Dia. For {1, 2, 3, 4, 6, 9} w.r.t. '|'



} Maximal elements

No upper bound

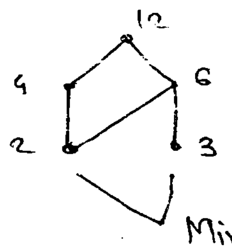
∴ Not lattice.

Lower bound is present.

GLB exists for every pair of elements

∴ It is meet semi-lattice.

Ex: Draw Hasse dia. for {~~1~~ 2, 3, 4, 6, 12} w.r.t. '|'



Not lattice.

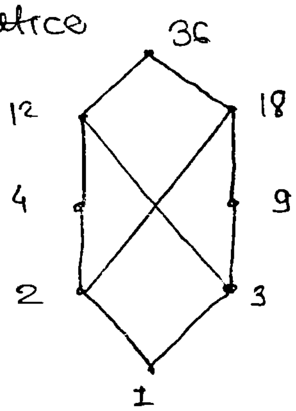
Minimal elements.

But it is join semi lattice

Ex: The poset {1, 2, 3, 4, 9, 12, 18, 36} w.r.t. '|' is ___

a) a join semi lattice b) a meet semi lattice

~~c)~~ a lattice d) none.



<u>Not lattice</u>

for 2 & 3 no LUB.

<u>whenever there are cross edges it is not lattice.</u>

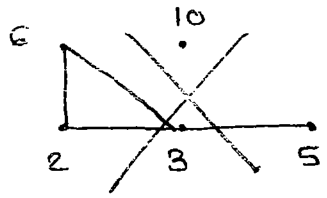
Ex: The poset $\{2, 3, 5, 6, 10, 15\}$ w.r.t. $|$ is

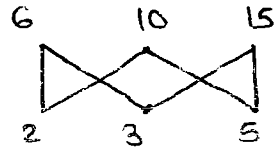
a) a join semi lattice but not a meet semi lattice

b) a meet semi lattice but not a join semi lattice

c) a lattice

~~d)~~ not a semi lattice





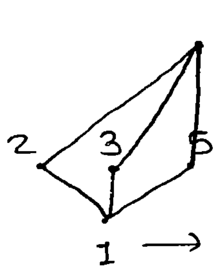
3 - Minimal

3 - Maximal element

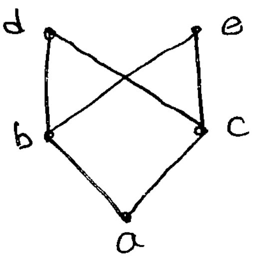
∴ it is not a lattice

Ex: The poset $\{1, 2, 3, 5, 30\}$ w.r.t. $|$

→ is a lattice

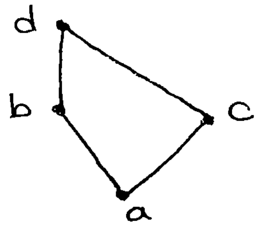


Ex: Consider the poset $P = \{a, b, c, d, e\}$ shown below which of the following is false?

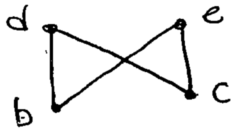


a) P is not a lattice.

b) The subset $\{a, b, c, d\}$ of P is a lattice.

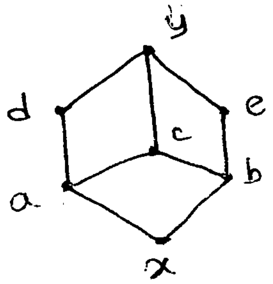


~~c)~~ The subset $\{b, c, d, e\}$ of P is a lattice.

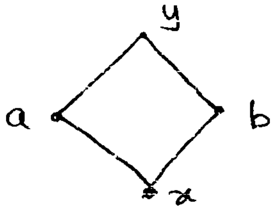


d) The subset $\{a, b, c, e\}$ of P is lattice.

Ex: Consider lattice $L = \{x, a, b, c, d, e, y\}$ shown below.
which of the following subsets of L are sub lattice of L.

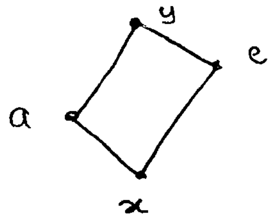


i) $L_1 = \{x, a, b, y\}$



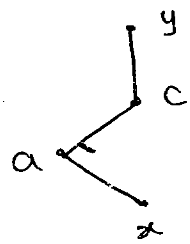
∴ Not sublattice because
LUB$_\vee$ in this lattice is y but in given diagram it is 'c'
of a & b

b) $L_2 = \{x, a, e, y\}$



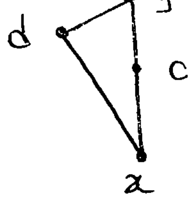
LUB of a & e is y
∴ L_2 is sublattice
for every pair of elements in L_2 LUB & GLB exists and they are same as original lattice.

c) $L_3 = \{x, a, c, y\}$



It is a lattice
L_3 is sublattice of L, because for element a & c LUB & GLB are same as L.

d) $L_4 = \{x, c, d, y\}$



Not sub lattice
because for d & c GLB is not same as given lattice.
In L it is 'a' and in this dia. it is 'x'.

Ex: which of the following is not true?

a) The upper bound of the lattice $[D_n; |]$ is n where n is a +ve integer.

b) The lower bound of the lattice $[D_n; |]$ is 1 where n is a +ve integer.

c) The ~~lower bound~~ of the lattice $[P(A); \subseteq]$ is ϕ where A is a finite set.

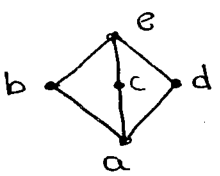
d) The upper bound of the lattice $[P(A); \subseteq]$ is A where A is a finite set.

~~e)~~ The upper bound of the lattice $[N; \leq]$ is ∞ where $N = \{1, 2, 3, \ldots \infty\}$.

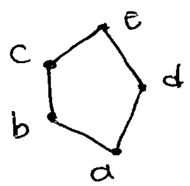
It is false because upper bound not exist and ∞ is not +ve integer

f) The lowerbound of the lattice $[N; \leq]$ is 1 where $N = \{1, 2, 3, \ldots \infty\}$

Ex: Which of the following is not a distributive lattice?



L_1^*



L_2^*

$$b \cup (c \cap d) = (b \cup c) \cap (b \cup d)$$
$$b \cup a \qquad e \cap e$$
$$b \neq e$$

L_1^* Not distributive

$\therefore$ ~~Not distributive lattice~~.

For L_2^*,

$$b \cup (c \cap d) = (b \cup c) \cap (b \cup d)$$
$$b \cup a \qquad c \cap e$$
$$b \neq c$$

$\therefore$ L_2^* also not distributive.

<u>Theorm</u>: A lattice L is not distributive iff L has a sublattice which is isomorphic to L_1^* or L_2^*.

Ex: which of the following lattice is NOT distributive?

a) $[P(A); \subseteq]$ where A is any finite set.

→ The elements of the power sets are sets. and for any 3 sets the distributive law holds good

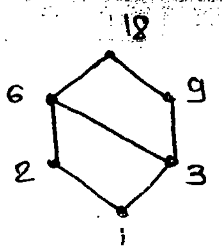
b) Every totally ordered set is a distributive lattice.

→ A totally ordered set can not have sublattice which is

c) [D_{18} ; |]

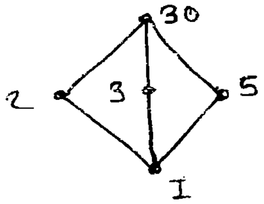
= { 1, 2, 3, 6, 9, 18 }

is a distributive lattice.



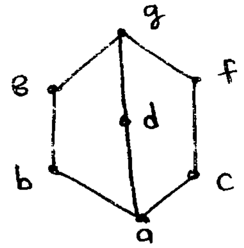
d) [{1, 2, 3, 5, 30} ; |]

∴ Not distributive.



$\equiv L_1^*$

Ex. A lattice L = { a, b, c, d, e, f } is shown below how many complements does the element 'b' has ?



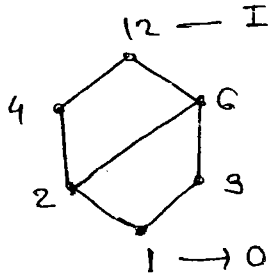
a) 1 b) 2 ~~c)~~ 3 d) 4

The complements of b are d, c, f.

The complements of d are 4 i.e. b, c, e, f.

<u>Note</u> : The given lattice is a complemented lattice but not a distributed lattice.

Ex : For the lattice [D_{12} ; |] , which of the following is not true ?



a) complement of 3 = 4

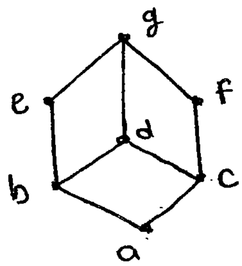
~~b)~~ ——— " ——— 2 = 6

c) ——— " ——— 1 = 12

d) ——— " ——— 6 does not exist.

It is distributive lattice but not complemented lattice.

Ex : For the lattice shown below which of the following is true ?



a) distributive but not complemented

b) Boolean algebra

c) complement but not distributive

→ In a lattice, 'd' has no complement.

∴ It is not complemented lattice.

'E' has two complements F & C

∴ It is not distributive.

Ex: If A is a finite set, then the poset [P(A) ; ⊆] is a boolean algebra.

→ If $x \in P(A)$, then

Complement of $x = A - x$

. If $A = \{a, b, c\}$ then the poset $\{P(A) ; \subseteq\}$ is a boolean algebra

complement of $\{a, c\} = A - \{a, c\}$

$= \{b\}$

<u>Square free integer</u> :- A positive integer n is said to be square free, if D_n has no elements that are perfect square (except 1).

Ex: If n is a square free integer then the poset $[D_n ; |]$ is a boolean algebra.

* If $[D_n ; |]$ is a boolean algebra then complement of

$$\bar{x} = \frac{n}{x} \quad \forall x \in D_n$$

Ex: which of the following Posets is not a boolean algebra.?

a) $[D_{21} ; |]$

$= [\{1, 3, 7, 21\} ; |]$

As 21 is square free integer

∴ D_{21} is boolean algebra.

b) $[D_{90} ; |]$

$90 \Rightarrow 2 \cdot 3 \cdot 3 \cdot 5$

(Not product of distinct prime integer)

∴ Not a square free integer

∴ D_{90} is not boolean algebra.

c) [D_{110} ; |]

$110 = 2 \cdot 5 \cdot 11$

$$\begin{array}{r|l} 2 & 110 \\ \hline 5 & 55 \\ \hline & 11 \end{array}$$

$\therefore$ square free integer.

$\therefore$ D_{110} is boolean algebra.

d) [D_{30} ; |]

$30 = 2 \cdot 3 \cdot 5$

D_{30} also boolean algebra.

Ex: In the boolean algebra [D_{110} ; |],

complement of $5 = \frac{110}{5}$

$$x = \frac{n}{x} \quad \forall\, x \in D_n$$

a) 10 b) 11 c) 22 d) 55

Groups :-

Algebraic structure : A non empty set S is called an algebraic structure w.r.t. a binary operation $*$, if $(a * b) \in S \quad \forall\, a, b \in S$

i.e. $*$ is a closed operation on S

$(S, *)$

$N = \{1, 2, 3, \dots \infty\}$

$Z =$ set of all integers

$Q =$ set of rational no.s

$R =$ set of all real no.s

Ex: $(N, +)$ is an algebraic structure.

Ex: $(N, \cdot)$ is an algebraic structure.

Ex: $(N, -)$ not an algebric structure.

Ex: $\{Z, +, \cdot, -\}$ an algebraic structure.

Ex: $\{Q, \div\}$ is not ——— 〃 ———.

Ex: $\{Q^*, \div\}$ is an algebraic structure

$Q^* = Q - \{0\}$ = set of all non zero rational no.

<u>Semi Group</u> :- An algebraic structure $(S, *)$ is called a semigroup iff $a*(b*c) = (a*b)*c$, $\forall$ $a, b, c \in S$

i.e. $*$ is associate on S.

$\Rightarrow$ $(Q^*, \div)$ is not a semi group because $\div$ is not associative

$$a \div (b \div c) \neq (a \div b) \div c$$

$\Rightarrow$ $(Z, -)$ is not a semi group

$$a-(b-c) \neq (a-b)-c$$

$\Rightarrow$ $(Z, +)$ is a semi group

$$a+(b+c) = (a+b)+c$$

$\Rightarrow$ $(N, \cdot)$ is a semigroup.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

<u>Monoid</u> :- A semi group $(S, *)$ is called a monoid if there exist an element $e \in S$, such that

$$a*e = e*a = a$$

i.e. e = Identity element of S w.r.t. $*$

Ex: $\{N, \cdot\}$ is a monoid, with identity element $1 \in N$

Ex: $\{N, +\}$ is not a monoid.

because $0 \notin N$.

Ex: $\{Z, +\}$ is a monoid

because $0 \in Z$

<u>Group</u> :- A monoid $(S, *)$ is called a group if to each element $a \in S$, there is an element $b \in S$, such that

$(a*b) = (b*a) = e$. ~~then~~

b = inverse of a

$= a^{-1}$

Ex: $(Z, +)$ is a group.

Ex : $(Z, \cdot)$ is not a group.

$$a^{-1} = \frac{1}{a}$$

$$a\left(\frac{1}{a}\right) = 1$$

Ex : $(Q, \cdot)$ is not a group

0 has no inverse

Ex : $(Q^*, \cdot)$ is a group

$$Q^* = Q - \{0\}$$

<u>Abelian Group</u> :- (Commutative group)

A group $(G, *)$ is said to be abelian, if $a*b = b*a \ \forall a,b \in G$

Ex : $(Z, +)$ is abelian group :- $a+b = b+a \ \forall a,b \in Z$

Ex : $(Q^*, \cdot)$ ——''—— . :- $(a \cdot b) = (b \cdot a) \ \forall a,b \in Q^*$

Ex : Set of all non singular matrices of order n is a group w.r.t. matrix multiplication but not a abelian group.

$$A \cdot B \neq B \cdot A$$

Ex : Set of all bijective functions on a set A is a group w.r.t. function composition but not abelian group.

$$(f \circ g) \neq (g \circ f)$$

Ex : Let $A = \{2, 4, 6, 8, \ldots \infty\}$
$B = \{1, 3, 5, 7, \ldots \infty\}$ which of the following is not a semigroup.

a) A w.r.t '+' b) A w.r.t '.'

c) B w.r.t. '+' d) B w.r.t. '.'

sum of two odd no. is even.

Ex : Let $A = \{1, 2, 3, \ldots \infty\}$ and $*$ denoted by $a*b = a^b$ then $(A, *)$ is

a) semigroup but not monoid

b) monoid but not a group

c) a group

d) not a semigroup.

$$(a*b)*c = (a^b)*c = (a^b)^c = a^{bc}$$

$$a*(b*c) = a*b^c = a^{b^c}$$

$$a^{bc} \neq a^{(b^c)}$$

Ex : Let $A = \{x \mid x$ is a real no. & $0 < x \leq 1\}$ then $(A, *)$ is

→ we have $a \cdot b \in A \ \forall \ a, b \in A$.

'·' is closed operation.

Associativity holds for any 3 real no.s betn 0 to 1

∴ '·' is associative.

1 is identity element w.r.t. '·' & $1 \in A$

but inverse of elements not exists on group except 1.

∴ Monoid but not a group

Ex : Let z = set of all integers & a binary operation * is defined as : $a * b = \max^m$ of $\{a, b\}$ then $(z, *)$ is —

→ we have $a * b = \max$ of $\{a, b\} \in z$, $\forall a, b \in z$

∴ * is closed on z.

also $(a*b)*c) = a*(b*c)$

$(3*4)*1 = 3*(4*1)$

$4*1 = 3*4$

$4 = 4$

∴ * is associative on z

$a*e = \max.$ of $\{a, e\} = a \quad \forall a, e \in z$

∴ The identity element w.r.t. * is the smallest integer which doesn't exists.

∴ $(z, *)$ is semi group.

Ex : Let Q^+ = set of all the +ve rational no.s & a binary operation * is defined as $(a*b) = \frac{ab}{3}$ then $(Q^+, *)$ is —

→ * is closed & associative on Q^+

Let e = identity element

$a*e = a \quad \forall a \in Q^+$

$\frac{ae}{3} = a$

$e = 3 \in Q^+$

Let a^{-1} = inverse of a

$a*a^{-1} = e \quad a \in Q^+$

$\frac{a*a^{-1}}{3} = e$

$a^{-1} = a, \ \in Q^+$

$\therefore (Q^+, *)$ is a group.

Properties :-

In a group $(G, *)$, the following properties hold good

1) The identity element e is unique.
2) The inverse of any element in G is unique.
3) ——— " ——————— identity element $'e' = e$
4) The cancellation laws hold good

$$(\cancel{a} * b) = (\cancel{a} * c) = b = c$$

$$(a * \cancel{c}) = (b * \cancel{c}) = a = b$$

5) $(a * b)^{-1} = b^{-1} * a^{-1}$

Ex: which of the following is not true ?

a) In a group $(G, *)$, if $a * a = a$ then $a = e$

b) In a group $(G, *)$, if $x^{-1} = x \ \forall\, x \in G$ then G is abelian.

consider, $(a * b)^{-1} = (b^{-1} * a^{-1}) \quad \forall\, a, b \in G$

$\therefore (a * b) = (b * a) \quad (\because x^{-1} = a)$

$\therefore G$ is abelian.

c) In a group $(G, *)$, if $(a * b)^2 = (a^2 * b^2)$ then G is abelian.

$\rightarrow$ Given that, $(a * b)^2 = (a^2 * b^2)$

$\Rightarrow (a * b) * (a * b) = (a * a) * (b * b)$

$$a * (b * a) * b = a * (a * b) * b$$

$$(b * a) = (a * b)$$

$\therefore G$ is abelian.

d) In a group $(G, *)$, if $(a * b)^n = (a^n * b^n)$ then G is abelian.

where $n = \{2, 3, 4 \ldots\}$ — TRUE

Finite Group :- A group with finite no. of elements.

Order of a group : $[O(G)]$ = No. of elements in G.

Ex : $\{0\}$ is a group w.r.t. '+'

— The only finite group of real no.s w.r.to '+' is $\{0\}$

Ex : $\{1\}$ is a group of order 1 w.r.to multipication.

Ex : $\{1,-1\}$ ——— " ——— 2 ——— " ———.

$\cdot$	1	-1
1	1	-1
-1	-1	1

⟶ Inverse of $1 = 1$

—— " —— $-1 = -1$

— The only finite group of real no. w.r.t. '$*$' are $\{1\}$ & $\{1,-1\}$

— In a group of order 2, $a^{-1} = a$, $\forall a \in G$

Ex : The cube root of unity $\{1, \omega, \omega^2\}$ is a group of order 3 w.r.t. '$*$'

$\circ$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$x^3 = 1$

$\omega^3 = 1$

$\omega^4 = \omega^3 \cdot \omega$

$= 1 \cdot \omega$

$\omega^4 = \omega$

Inverse of $1 = 1$ $(\because 1 \times 1 = 1)$

Inverse of $\omega = \omega^2$ $(\because \omega \times \omega^2 = 1)$

Inverse of $\omega^2 = \omega$ $(\because \omega^2 \times \omega = 1)$

$\therefore$ It is group of order 3.

Ex : The set $\{1, -1, i, -i\}$ is a group w.r.to. multipication.

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Inverse of $1 = 1$

—— " —— $-1 = -1$

—— " —— $i = -i$

—— " —— $-i = i$

$\therefore$ It is group of order 4.

Addition Modulo M, $\oplus_M$

If a and b are any +ve integers then $a \oplus_m b$ defined as

$$a \oplus_M b = \begin{cases} (a+b) & \text{if } (a+b) < m \\ r & \text{if } (a+b) \geq m \text{ where } r \text{ is remainder obtained by dividing } (a+b) \text{ with } m. \end{cases}$$

Ex : $m = 6$

$2 \oplus_6 3 = 5$ $\quad 4 \oplus_6 4 = 2$

$3 \oplus_6 4 = 1$

Ex : $\{0, 1, 2, --- m-1\}$ is a group w.r.t. $\oplus_M$

Multipication Modulo M, $\otimes_M$:

If a & b are any two +ve integers then

$$a \otimes_m b = \begin{cases} (ab) & \text{if } (ab) < m \\ r & \text{if } (ab) \geq m \text{ where } r \text{ is remainder obtained by dividing } (ab) \text{ withm } m. \end{cases}$$

Ex : $2 \otimes_7 3 = 6$

Ex : $5 \otimes_7 3 = 1$

Ex : $\{1, 2, 3 \cdots p-1\}$ is a group w.r.to $\otimes_p$ where p is a prime no.

* If n is a +ve integer, then S_n = set of all +ve integers which are less than n and relatively prime to n.

$S_6 = \{1, 5\}$

$S_8 = \{1, 3, 5, 7\}$, $S_7 = \{1, 2, 3, 4, 5, 6\}$

* If n is any +ve integer then S_n is a group w.r.to $\otimes_n$

Ex : The set $\{0, 1, 2, 3, 4, 5\}$ is a group w.r.t. $\oplus_6$ which of the following is NOT TRUE ?

(a) The inverse of $1 = 5$

(b) —— " —— $2 = 4$

(d) —— " —— $4 = 2$

$1 \oplus_6 5 = 0$ but $3 \oplus_6 0 \neq 0$

* In a group of even order i.e. group of even no. of element there exists atleast one element a $(a \neq e)$ such that $a^{-1} = a$

Ex : The set $\{1, 2, 3, 4, 5, 6\}$ is group w.r.t. $\otimes_7$ which is False ?

ⓐ The inverse of $1 = 1$ $1 \otimes_7 1 = 1 \rightarrow$ Identity element.

ⓑ —— " —— $2 = 4$ $2 \otimes_7 4 = 1$

ⓒ —— " —— $3 = 5$ $3 \otimes_7 5 = 1$

ⓓ —— " —— $6 = 2$ $6 \otimes_7 2 \neq 1$

Ex : The set $\{1, 3, 5, 7\}$ $\otimes_8$ which is False ?

ⓐ The inverse of $1 = 1$ $1 \otimes_8 1 = 1$

ⓑ —— " —— $3 = 7$ $7 \otimes_8 3 \neq 1$

ⓒ —— " —— $5 = 5$ $5 \otimes_8 5 = 1$

ⓓ —— " —— $7 = 7$ $7 \otimes_8 7 = 1$

Ex : Consider a set $G = \{2, 4, 6, 8\}$ w.r.t. $\otimes_{10}$ which is false ?

ⓐ G is group w.r.t. $\otimes_{10}$

ⓑ The identity element of G is 6

ⓒ The inverse of $2 = 4$

ⓓ —— " —— $4 = 4$

$\otimes_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

The inverse of $2 = 8$

—— " —— $4 = 4$

—— " —— $6 = 6$

—— " —— $8 = 2$

Ex : which of the following is ~~not true~~? a group ?

a) $\{1, 2, 3, 4, 5\}$ w.r.t. $\otimes_6$

$2 \otimes_6 3 = 0 \notin A$

closure property fails

$\therefore$ Not a group.

b) $\{0, 1, 2, 3, 4, 5\}$ w.r.t. $\otimes_6$

For '0' no multiplicative inverse exists

$0 \otimes_6 x = 0$

$\therefore$ Not a group.

c) $\{1, 2, 3, 4, 5\}$ w.r.t. $\oplus_6$

$2 \oplus_6 4 = 0$

$\therefore$ Not a group.

d) $\{1, 2, 3, 4\}$ w.r.t $\otimes_5$

It is a group.

Order of an element of a group :-

Let $(G, *)$ be a group with identity element 'e'

for $a \in G$.

The order of $a = O(a)$ is defined as n

where n is smallest +ve integer such that $a^n = e$

Ex: In the group $\{1, -1\}$ w.r.t. multipication.

$O(1) = 1$
$O(-1) = 2$

Ex: In the group $\{1, \omega, \omega^2\}$ w.r.t. multipication.

$O(1) = 1$

$O(\omega) = 3$

$O(\omega^2) = 3$

Ex: In the group $\{1, \omega, \omega^2\}$ w.r.t. multipication

* order of a = order of a^{-1} $\forall a \in G$.

Ex: In the group $\{1, -1, i, -i\}$ w.r.t multipication

$O(1) = 1$ $\quad$ $O(i) = 4$

$O(-1) = 2$ $\quad$ $O(-i) = 4$

* In a group $(G, *)$ order of a = divisor of $O(G)$ $\forall a \in G$.

Ex: In the group $\{0, 1, 2, 3\}$ w.r.t. $\oplus_4$,

order of $0 = 1$
order of $1 = 4$
order of $2 = 2$
order of $3 = 4$

$2^2 = 2 \oplus_4 2 = 0$

$1^3 = (1 \oplus_4 1) \oplus_4 1$
$= 3$

$1^4 = 1^3 \oplus_4 1$
$= 3 \oplus_4 1$

$1^4 = 0$

Ex: Group $\{1, 3, 5, 7\}$ w.r.t. $\otimes_8$

$O(1) = 1$ $\quad\quad 3^2 = 3 \otimes_8 3 = 1$

$O(3) = 2$

$O(5) = 2$

$O(7) = 2$

<u>Subgroups</u> :- Let $(G, *)$ be a group. A subset H of G is called a subgroup of G, if H is a group w.r.t. $*$

Ex: $G = \{1, -1, i, -i\}$ is a group w.r.t. multipication

$H_1 = \{1, -1\}$ is a subgroup of G.

$H_2 = \{1\}$ is a subgroup.

<u>Note</u> :- For the group $(G, *)$ with identity element e, G and $\{e\}$ are called trivial subgroup of G.

<u>Proper Subgroup</u> :- Any subgroup of $(G, *)$, which is not a trivial subgroup of G, is called as proper subgroup of G.

<u>Theorm 1</u> : Let H be a non empty subset of a group $(G, *)$ H is a subgroup of G iff $(a * b^{-1}) \in H \quad \forall a, b \in H$

<u>Theorm 2</u> : Let H be a non empty finite subset of a group $(G, *)$ then H is a subgroup of G iff $(a * b) \in H \quad \forall a, b \in H$

<u>Theorm 3</u> : (Lagrange's theorm)

If H is a subgroup of a group $(G, *)$. Then $O(H)$ is a divisor of $O(G)$.

Ex: $G = \{1, 2, 3, 4, 5, 6\}$ is a group w.r.t. $\otimes_7$

which of the following are subgroup of G ?

a) $\{1, 6\}$ $\quad$ c) $\{1, 2, 4\}$

b) $\{1, 3, 5\}$ $\quad$ d) $\{1, 3\}$

	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2

Ex: $G = \{0, 1, 2, 3, 4, 5\}$ is a group w.r.t. $\oplus_6$

which are subgroups?

(a) $\{0, 3\}$ (c) $\{0, 2, 4\}$

(b) $\{0, 1, 5\}$ d) $\{0, 1, 2, 4, 5\}$

Ex: which of the following stat. is not true?

a) Every subgroup of an abelian group is also an abelian group.

(b) Union of any two subgroups of a group 'G' is also a subgroup of G.

c) Intersection —— " ——.

d) Union of 2 subgroups H_1 & H_2 of a group $(G, *)$ is also a subgroup of G, iff $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Ex: If $(G, *)$ is a group of order P, where P is a prime no. then no. of subgroups in $(G, *)$ is ——

a) $P-1$

b) P

(c) 2

d) 0

→ Let $(H, *)$ be a subgroup of G.

Let $O(H) = n$

By lagranges theorm n is ~~divisible~~ divisor by p.

$\Rightarrow n = 1$ or $n = p$

$\Rightarrow H = \{e\}$ or $H = G$

<u>Cyclic Group</u> :- Let $(G, *)$ be a group with identity element e. If there exists an element $a \in G$, such that every element of G can written in the form a^n for some integer n. then G is called as cyclic group. and element a is called Generator of G.

Ex: The set $\{1, -1\}$ is a cyclic group with respect to multipication generator -1.

$$\{1, -1\} \quad 1 = (-1)^2, \; -1 = (-1)^1$$

Ex: $\{1, \omega, \omega^2\}$ w.r.t. Multi.

Generator $= \omega, \omega^2$

* In a cyclic group $(G, *)$ with a generator a, a^{-1} is also a generator of G.

Ex: The set $\{1, -1, i, -i\}$ is a cyclic group w.r.t. multiplication.

The generators $= i, -i$

* $\boxed{O(G) = O(a)}$ (* Identity elements can not be generator.)

Ex: The set $\{0, 1, 2, 3\}$ is a cyclic group w.r.t. $\oplus_4$

$$\left.\begin{aligned} 1^2 &= 1 \oplus_4 1 = 2 \\ 1^3 &= 1^2 \oplus_4 1 = 2 \oplus_4 1 = 3 \\ 1^4 &= 1^3 \oplus_4 1 = 3 \oplus_4 1 = 0 \end{aligned}\right\} \text{ 1 is Generator.}$$

Generator is 1 & 3.

3 is inverse of 1.

Ex: The set $\{1, 2, 3, 4\}$ is a cyclic group w.r.t. $\otimes_5$

$$\left.\begin{aligned} 2^2 &= 2 \otimes_5 2 = 4 \\ 2^3 &= 2^2 \otimes_5 2 = 4 \otimes_5 2 = 3 \\ 2^4 &= 2^3 \otimes_5 2 = 3 \otimes_5 2 = 1 \end{aligned}\right\} \text{ 2 is Generator.}$$

As 3 is inverse of 2

$\therefore$ 3 is also generator.

* Let $(G, *)$ be a cyclic group of order n, with generator a.
  i) a^m is also a generator of G, if m and n are coprime
  i.e. $\gcd(m, n) = 1$
  ii) Number of generators in G $= \phi(n) =$ Euler function of n.
  $=$ no. of positive integers less than n and relatively prime to n.

Ex: Let $(G, *)$ be a cyclic group of order 8 with generator a then how many generators are there and what are they?

$\rightarrow$ No. of generator $= \phi(8)$ $(\because \{1, 2, 3, 4, 5, 7\})$

$= 4$

The generators are $= a^1, a^3, a^5, a^7$.

Ex: Let $(G, *)$ be a cyclic group of order 70 then no. of Generator in G = ?

$\rightarrow$ $\phi(70) = 70\left\{\dfrac{(2-1)(5-1)(7-1)}{2 \cdot 5 \cdot 7}\right\}$

$$\begin{array}{r|l} 2 & 70 \\ \hline 5 & 35 \\ \hline & 7 \end{array}$$

* $\phi(n) = n\left[\dfrac{(P_1-1)(P_2-1)\cdots(P_k-1)}{P_1\cdot P_2\cdots P_k}\right]$ where $P_1, P_2, P_3 \cdots P_k$ distinct prime factors of n.

Ex: $\{0,1,2,3,4\}$ is a cyclic group of order 5 then how many generators are there & what are they? (w.r.t. $\oplus_5$)

→ $\phi(n) = n-1$ — if n - prime no.

$\phi(5) = 4$ & The generators are 1, 2, 3, 4

Ex: The set $\{1,2,3,4\}$ is a cyclic group w.r.t. $\otimes_7$ of order 6 then how many generator & what are they?

→ $\therefore \phi(6) = 2$ $\{1,2,3,4,5,6\}$

$2^2 = 2 \otimes_7 2 = 4$

$2^3 = 2^2 \otimes_7 2 = 1$

$\Rightarrow O(2) = 3$

$3^2 = 3 \otimes_7 3 = 2$

$3^3 = 3^2 \otimes_7 3 = 2 \otimes_7 3 = 6$

$3^5 = 3^4 \otimes_7 3 = 4 \otimes_7 3 = 5$

$3^6 = 3^5 \otimes_7 3 = 5 \otimes_7 3 = 1$

$\therefore$ 3 is a generator

Also 5 is generator as inverse of 3.

Ex: The group $\{1,3,5,7\}$ w.r.t. $\otimes_8$ is NOT a cyclic group. there is no generator for this group.

$1^2 = 1 \otimes_8 1 = 1$

$3^2 = 3 \otimes_8 3 = 1$

* Following properties hold good in cyclic group
  1. Every group of prime order is cyclic.
  2. Every ~~ell~~ cyclic group is abelian group.
  3. every group of prime order is abelian group.
  4. Every subgroup of a cyclic group is cyclic.

Functions :-

- A relation F from A to B, is called a function, if each element of A is mapped with a unique element in B. & denoted by $F: A \to B$.

  Here $A \to$ Domain of F and $B \to$ co-domain of F.
- Range of $F = \{y \mid y \in B$ and $(x, y) \in F\}$
- Range of $F \subseteq$ co-domain of F.
- If Range = co-domain, then it is an 'onto' function.
- A function $F: A \to A$ is called a function on A.
- If $|A| = m$ and $|B| = n$, then no. of functions possible from A to B $= n^m$.

One to one function (Injection) :-

- A function $f: A \to B$ is said to be one to one, if no two elements in A have same image in B. or if $f(a) = f(b) \Rightarrow a = b$

* If A and B are finite sets then a 1 to 1 function from A to B is possible iff no. of elements in A i.e. $|A| \leq |B|$

* If $|A| = m$ & $|B| = n$ then no. of 1 to 1 functions possible from A to B $(m \leq n)$ ~~then~~ $= P(n, m) = {}^nP_m$

If $|A| = |B| = n$ then $= {}^nP_n = n!$

On-to function

- A function $F: A \to B$ is said to be on-to if each element of B is mapped by atleast one element of A. i.e. Range of F = B = codomain.
- If A and B are finite sets, then an onto function from A to B is possible iff $|B| \leq |A|$
- If $|A| = m$ and $|B| = n$ $(n < m)$ then no. of onto functions possible $= n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 (n-3)^m + \cdots$

$${}^nC_{n-1} (-1)^n \, 1^m$$

Ex: For $m = 6$ $n = 3$

$= 3^6 - {}^3C_1 \cdot 2^6 + {}^3C_2 \cdot 1^6$

$= 729 - 3(64) + 3$

$= 5\ldots$

Note :- If $|A| = |B|$ then every 1 to 1 funⁿ from $A \to B$ is onto and vice versa.

- If $|A| = |B| = n$ then no. of onto functions $= n!$

Bijection :- A function $F: A \to B$ is said to be a bijection, if $F: A \to B$ is 1 to 1 and onto

* If A and B are finite sets then a Biejection from A to B is possible iff $|A| = |B|$
* If $|A| = |B| = n$ then no. of bijection possible from A to B $= n!$

Inverse of a function :- If $F: A \to B$, then the inverse relation from B to A, is called a function (if the inverse relⁿ F^{-1} from B to A is a function), then it is called inverse of F, and denoted by $F^{-1}: B \to A$

* Inverse of a function $F: A \to B$ exists iff $F: A \to B$ is a bijection.

Identity function :- $F: A \to B$ is called an identity function if $f(x) = x \quad \forall x \in A$

$I: A \to A$ or I_A.

- Every identity function is a bijection.

Function composition :-

$F: A \to A$ and $g: A \to A$, then

i) $(F \circ g): A \to A$ defined by $(F \circ g)(x) = F\{g(x)\}$

ii) $(g \circ F): A \to A$ defined by $(g \circ F)(x) = g\{F(x)\}$

- In general $(F \circ g) \neq (g \circ f)$
- If $F: A \to A$ and $I: A \to A$ then

$(F \circ I) = (I \circ F) = F$

- If $F: A \to A$ is a bijection, then $(F \circ F^{-1}) = (F^{-1} \circ F) = I$
- If $F: A \to B$ and $g: B \to C$, then

i) $(g \circ F): A \to C$

ii) $(F \circ g)$ may not be defined

$(g \circ F): A \to A$ defined by

only when ~~range of g~~ $\subseteq$ domain of F.

- If $F: A \to B$ then i) $F \circ I_A = f$ and
  ii) $I_B \circ F = F$

- If $F: A \to B$ is a bijection then i) $F \circ F^{-1} = I_B$ and $F^{-1} \circ F = I_A$

$F^{-1} = B \to A$

Ex: which of the following is a function on set of all real no.s

a) $F_1(x) = 1/x$ → for the real no. '0' image does not exists.

b) $F_2(x) = \sqrt{x}$ → For -ve real no. in domain image not exists

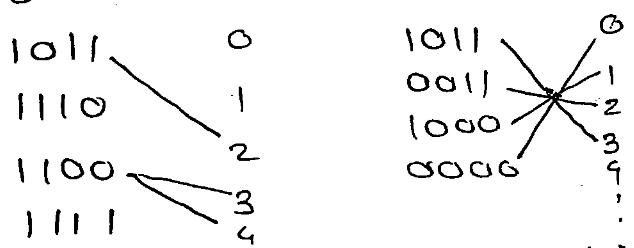
c) $F_3(x) = \pm\sqrt{x^2+1}$ —— For $1 < {-\sqrt{2} \atop \sqrt{2}}$ } two images of each element ∴ Not function

d) $F_4(x) = |x|$ $\quad {-1 \atop 1} > 1$ (For two element one image)
∴ It is a Function.

Ex: which of the following relations from set of all bit strings to set of all integers is a function

$F_1(S)$ = The position of a zero bit in the bit string S
$F_2(S)$ = The no. of 1 bits in the bit string S.

→ $F_1 : S^* \to Z$ $\qquad F_2 : S^* \to Z$



f_1 is not a function because a string with 2 or more '0' will have two or more images

For any bit string the no. of 1 bits is non-ve integer
∴ Each bit string has only one image.
∴ F_2 is a function.

Ex: Consider the following function on set of all integers.

$$f(x) = x^3 \text{ and } g(x) = \left\lceil \frac{x}{2} \right\rceil$$

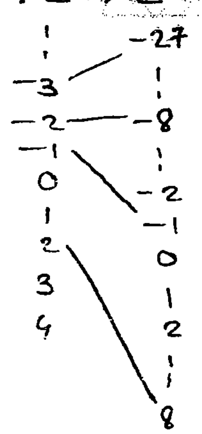
which of the following stat. are true ?

S_1) F is 1 to 1

S_2) F is onto

S_3) g is 1 to 1

→ $f : Z \to Z$



let, $f(a) = f(b)$

$a^3 = b^3$

$a = b$

$\therefore$ f is 1 to 1

f is not onto.

g is not 1 to 1

bcoz the integer 1 & 2 in domain has same image.

g is onto function.

ex. Let $A = R - \{3\}$ and $B = R - \{1\}$ where $f : A \to B$ is defined by $f(x) = \left(\frac{x-2}{x-3}\right)$ which is true ?

a) f is 1-1 but not onto

b) f is onto but not 1-1

c) f is a bijection

d) f is neither 1-1 nor onto

→ $f(a) = f(b)$

$$\frac{a-2}{a-3} = \frac{b-2}{b-3}$$

$$(a-2)(b-3) = (b-2)(a-3)$$

$$a = b$$

$\therefore$ one to one function.

Let $f(x) = \frac{x-2}{x-3} = y$

$$x - 2 = (x-3)\,y$$

$$x = \frac{2-3y}{1-y}$$ — inverse of f(x)

Check for inverse is a function (or) not. if inverse is a function for each value of real no. except 1

$\therefore$ f is onto

$\therefore$ f is bijection.

Ex : For which of the following function inverse is not defined on their range.

A. $f(x) = x^2$ —— Not one to one $\therefore$ No inverse exist

B. $f(x) = x^3$ —— one to one & onto $\therefore$ inverse exists

$$f^{1}(x) = x^{1/3}$$

C. $F_1(x) = \sin x$, $x \in (0, \pi)$ —— Not one to one $\therefore$ inverse not exists.

$$\left.\begin{matrix}\pi/4 \\ 3\pi/4\end{matrix}\right\rangle \ 1/\sqrt{2} \quad (\text{Two images})$$

D. $F(x) = 2^x$

$$2^x = y$$

$$x.\log e = \log y$$

$$x = \log_2 y$$

$$F^{1}(x) = \log_2 x \quad (\text{Inverse})$$