Chapter 1

Measurable Sets

1.1 Introduction

This chapter begins the development of Lebesgue integration, which constitutes Part I of the text. The theory may be seen as arising from the need to overcome some of the shortcomings of the Riemann integral, which is restrictive in both the kind of function that may be integrated and the space over which the integration takes place. These shortcomings make the Riemann integral unsuitable for certain applications, for example those involving random parameters. A further complication with the Riemann theory concerns the integration of a pointwise limit of a sequence of Riemann integrable functions, such limits sometimes failing to be Riemann integrable. The removal of these limitations may be seen as a reason for the wide applicability of the Lebesgue theory.

Nevertheless, the Riemann integral still occupies an important position in analysis. Indeed, as we shall see, the set of Lebesgue integrable functions on $[a, b]$ is the completion in a precise sense of the set of Riemann integrable functions, much as the real number system is the completion of the rational number system.

It is illuminating to compare the construction of the two integrals in terms of how the domain $[a, b]$ of an integrand f is partitioned. In the case of the Riemann integral, $[a, b]$ is partitioned into subintervals $[x_{i-1}, x_i]$ and a point x_i^* is chosen in each. A suitable limit of the corresponding Riemann sums $\sum_i f(x_i^*)\Delta x_i$ then produces the Riemann integral of f. By contrast, in the Lebesgue theory it is the *range* of the function that is partitioned into subintervals, these inducing, via preimages under f, a partition of $[a, b]$. This partition will in general *not* consist of intervals. However, the Lebesgue theory provides a way of "measuring" the members of the partition. The Lebesgue integral is then constructed by multiplying these measured values by (approximate) function values, summing, and taking limits.

The preceding discussion suggests (correctly) that a fundamental feature of the Lebesgue theory is the notion of "measure" of a set. Such measures are constructed by starting with a collection $\mathcal{A}$ of elementary sets, such as intervals in $\mathbb{R}$ or rectangles in $\mathbb{R}^2$, and a *set function* that assigns a natural "size" to each member of $\mathcal{A}$, for example length in the case of intervals and area in the case of rectangles. The collection $\mathcal{A}$ is then enlarged to a richer class of sets that can still be "measured," the so-called *σ-field of measurable sets*. Unlike $\mathcal{A}$, this collection is closed under standard set-theoretic operations, including countable unions and intersections, a feature eventually resulting in limit theorems of a sort unavailable in Riemann integration, these theorems underlying much of modern analysis. The first step then in the construction of the Lebesgue integral is to develop the notion of measurable set and measure, which is the goal of this chapter.

1.2 Measurable Spaces

For a robust theory of integration that admits the standard combinatorial and limit operations, one requires that the collections of measurable sets on which the integration is based be closed under the usual set-theoretic operations. In this section we discuss the most common of such collections.

Fields and Sigma Fields

Let X be a nonempty set. A **field** on X is a family $\mathfrak{F}$ of subsets of X satisfying (a)–(c) of the following. If $\mathfrak{F}$ also satisfies (d), then $\mathfrak{F}$ is called a σ-**field**:

(a) $X \in \mathfrak{F}$.

(b) $A \in \mathfrak{F} \Rightarrow A^c \in \mathfrak{F}$.

(c) $A,\, B \in \mathfrak{F} \Rightarrow A \cup B \in \mathfrak{F}$.

(d) $A_1,\, A_2, \cdots \in \mathfrak{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{F}$.

Note that (a) and (b) imply that $\emptyset \in \mathfrak{F}$. An induction argument using (c) shows that a field $\mathfrak{F}$ is closed under finite unions, that is,

$$A_1, \ldots, A_n \in \mathfrak{F} \Rightarrow A_1 \cup \cdots \cup A_n \in \mathfrak{F}.$$

Of course, every field with only finitely many members is a σ-field, since in this case countable unions reduce to finite unions. De Morgan's law

$$A_1 \cap A_2 \cap \cdots \cap A_n = \left(A_1^c \cup A_2^c \cup \cdots \cup A_n^c\right)^c$$

together with (b) shows that a field is closed under finite intersections and thus, for example, under the operation of **symmetric difference** defined by

$$A \triangle B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

Furthermore, every finite union of members of a field may be expressed as a *disjoint* union of members of the field via the construction

$$\bigcup_{k=1}^{n} A_k = A_1 \cup (A_2 \cap A_1^c) \cup \cdots \cup (A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c). \tag{1.1}$$

Similar remarks apply to σ-fields: Part (d) of the above definition asserts that a σ-field is closed under countable unions, and an application of De Morgan's law shows that a σ-field is closed under countable intersections as well. As a consequence, a σ-field $\mathfrak{F}$ is closed under the operations of **limit infimum** and **limit supremum** defined, respectively, by

$$\varliminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \quad \text{and} \quad \varlimsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Moreover, every countable union of members of $\mathfrak{F}$ may be expressed as a countable *disjoint* union of members of $\mathfrak{F}$ in the manner of (1.1):

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_2 \cap A_1^c) \cup \cdots \cup (A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c) \cup \cdots . \tag{1.2}$$

Members of a σ-field $\mathfrak{F}$ on X are called **$\mathfrak{F}$-measurable sets**. The qualifier $\mathfrak{F}$ is usually dropped if the σ-field is understood and there is no possibility of confusion. The pair $(X, \mathfrak{F})$ is called a **measurable space**. A finite or countably infinite sequence of disjoint measurable sets with union A is called a **measurable partition** of A.

1.2.1 Examples.

(a) The power set $\mathcal{P}(X)$ is obviously a σ-field, as is the collection $\{\emptyset, X\}$. A field clearly cannot have exactly three members. All fields with exactly four members are of the form $\{\emptyset, X, A, A^c\}$.

(b) A subset A of X is said to be **cofinite** if A^c is finite. The collection $\mathcal{F}$ of all sets that are either finite or cofinite is a field. If X is infinite, then $\mathcal{F}$ is not a σ-field (Ex. 1.9).

(c) A subset A of X is said to be **cocountable** if A^c is countable. The collection $\mathcal{F}$ of all sets that are either countable or cocountable is a σ-field. For example, to see that $\mathcal{F}$ is closed under countable unions $A = \bigcup_{n=1}^{\infty} A_n$, note that if each A_n is countable, then A is countable and if some A_n is cocountable then A is cocountable. In either case, $A \in \mathcal{F}$.

(d) If $\mathcal{F}$ is a field (σ-field) on X, then the trace

$$\mathcal{F} \cap E = \{A \cap E : A \in \mathcal{F}\}$$

is a field (σ-field) on E. For example, if $A,\ B \in \mathcal{F}$, then the relations

$$(A \cup B) \cap E = (A \cap E) \cup (B \cap E) \quad \text{and} \quad (A \setminus B) \cap E = (A \cap E) \setminus (B \cap E)$$

show that $A \cup B,\ A \setminus B \in \mathcal{F}$. Note that $\mathcal{F} \cap E \subseteq \mathcal{F}$ iff $E \in \mathcal{F}$, in which case $\mathcal{F} \cap E$ is simply the collection of all sets $A \in \mathcal{F}$ with $A \subseteq E$. $\Diamond$

Generated Sigma Fields

The intersection of a nonempty family of σ-fields on a nonempty set X is easily seen to be a σ-field. In particular, if $\mathcal{A}$ is an arbitrary nonempty collection of subsets of X, then the intersection $\sigma(\mathcal{A})$ of all σ-fields on X containing $\mathcal{A}$ is a σ-field, called **σ-field generated by $\mathcal{A}$**. Note that there is at least one σ-field containing $\mathcal{A}$, namely, $\mathcal{P}(X)$, hence $\sigma(\mathcal{A})$ is well-defined. Generated σ-fields have the important **minimality property**:

$$\mathcal{F} \textit{ a } \sigma\textit{-field and } \mathcal{A} \subseteq \mathcal{F} \Rightarrow \sigma(\mathcal{A}) \subseteq \mathcal{F}.$$

The **field generated by $\mathcal{A}$**, denoted by $\varphi(\mathcal{A})$, is defined in a similar manner and enjoys the analogous minimality property.

1.2.2 Example. Let $\mathcal{A} = \{A_1, A_2, \ldots\}$ be a countable partition of X. Then $\sigma(\mathcal{A})$ consists of all unions $\bigcup_{n \in S} A_n$, where $S \subseteq \mathbb{N}$. (If $S = \emptyset$, then the union is defined to be $\emptyset$.)

To see this, note first that the collection $\mathcal{F}$ of all such unions is a σ-field. Indeed, $\mathcal{F}$ is obviously closed under countable unions, and by disjointness

$$\left(\bigcup_{n \in S} A_n\right)^c = \bigcup_{n \in S^c} A_n,$$

hence $\mathcal{F}$ is closed under complements as well. Since $\mathcal{A} \subseteq \mathcal{F} \subseteq \sigma(\mathcal{A})$, the minimality property implies that $\sigma(\mathcal{A}) = \mathcal{F}$. The analogous assertions hold for finite partitions of X. $\Diamond$

Borel Sets

Let X be a topological space. The σ-field generated by the collection of all open subsets of X is called the **Borel σ-field on** X and is denoted by $\mathcal{B}(X)$. A member of $\mathcal{B}(X)$ is called a **Borel set**. The minimality property of $\mathcal{B}(X)$ takes the following form:

If a σ-field $\mathcal{F}$ contains all open sets, then it contains all Borel sets.

Borel σ-fields provide a bridge between topology and measure theory, allowing, for example, the entry of continuous functions into integration theory.

Since closed sets are complements of open sets, $\mathcal{B}(X)$ is also generated by the collection of closed sets. For Euclidean space $\mathbb{R}^d$, more can be said:

1.2.3 Proposition. *The σ-field $\mathcal{B}(\mathbb{R}^d)$ is generated by the collection*

(a) $\mathcal{O}_I$ *of all bounded, open d-dimensional intervals* $(a_1, b_1) \times \cdots \times (a_d, b_d)$.

(b) $\mathcal{C}_I$ *of all bounded, closed d-dimensional intervals* $[a_1, b_1] \times \cdots \times [a_d, b_d]$.

(c) $\mathcal{H}_I$ *of all bounded, left-open d-dimensional intervals* $(a_1, b_1] \times \cdots \times (a_d, b_d]$.

Proof. For ease of notation we prove the proposition for $d = 1$; the proof for the general case is entirely similar.

(a) Let $\mathcal{O}$ denote the collection of all open sets in $\mathbb{R}$. Since $\mathcal{O}_I \subseteq \mathcal{O}$, by minimality we have $\sigma(\mathcal{O}_I) \subseteq \sigma(\mathcal{O}) = \mathcal{B}(\mathbb{R})$. On the other hand, every member of $\mathcal{O}$ is a countable union of sets in $\mathcal{O}_I$, hence $\mathcal{O} \subseteq \sigma(\mathcal{O}_I)$ and so $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{O}_I)$.

(b) Let $\mathcal{C}$ denote the collection of all closed sets in $\mathbb{R}$. As in part (a), $\sigma(\mathcal{C}_I) \subseteq \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$. Moreover, every bounded open interval (a, b) may be expressed as $\bigcup_n [a + 1/n, b - 1/n]$, hence $\mathcal{O}_I \subseteq \sigma(\mathcal{C}_I)$. By part (a) and minimality, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}_I) \subseteq \sigma(\mathcal{C}_I)$.

(c) From the representations $(a, b) = \bigcup_n (a, b - 1/n]$ and $(c, d] = \bigcap_n (c, d + 1/n)$, we see that $\mathcal{O}_I \subseteq \sigma(\mathcal{H}_I)$ and $\mathcal{H}_I \subseteq \sigma(\mathcal{O}_I)$. By minimality, $\sigma(\mathcal{O}_I) \subseteq \sigma(\mathcal{H}_I)$ and $\sigma(\mathcal{H}_I) \subseteq \sigma(\mathcal{O}_I)$. An application of (a) completes the argument. $\square$

The collection $\mathcal{H}_I$ will figure prominently in the development of the Lebesgue integral on Euclidean space $\mathbb{R}^d$.

Extended Borel Sets

To deal with functions that take values in $\overline{\mathbb{R}}$, we need to augment $\mathcal{B}(\mathbb{R})$ with the sets

$$B \cup \{-\infty\}, \quad B \cup \{\infty\}, \quad B \cup \{-\infty, \infty\}, \quad B \in \mathcal{B}(\mathbb{R}).$$

The collection of all such sets, together with the Borel subsets of $\mathbb{R}^d$, is called the **extended Borel σ-field** and is denoted by $\mathcal{B}(\overline{\mathbb{R}})$. One easily checks that $\mathcal{B}(\overline{\mathbb{R}})$ is indeed a σ-field with trace $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$. It may be shown that $\overline{\mathbb{R}}$ has a natural topology whose open sets generate $\mathcal{B}(\overline{\mathbb{R}})$ (Exercise 2.30).

Product Sigma Fields

Let $X_1, \ldots, X_d$ be nonempty sets and set $X := X_1 \times \cdots \times X_d$. For arbitrary nonempty collections $\mathcal{A}_j \subseteq \mathcal{P}(X_j)$ define

$$\mathcal{A}_1 \times \cdots \times \mathcal{A}_d = \{A_1 \times \cdots \times A_d : A_j \in \mathcal{A}_j,\ j = 1, \ldots, d\}.$$

If $\mathcal{F}_j$ is a σ-field on X_j, then the σ-field on X generated by $\mathcal{F}_1 \times \cdots \times \mathcal{F}_d$ is called the **product σ-field** and is denoted by $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_d$. Thus

$$\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_d := \sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_d).$$

Members of $\mathcal{F}_1 \times \cdots \times \mathcal{F}_d$ are called **measurable rectangles**.

1.2.4 Theorem. *If $\mathcal{A}_j \subseteq \mathcal{P}(X_j)$, then*

$$\sigma(\mathcal{A}_1) \otimes \cdots \otimes \sigma(\mathcal{A}_d) = \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d). \tag{1.3}$$

Proof. The inclusion $\supseteq$ follows from $\sigma(\mathcal{A}_1) \otimes \cdots \otimes \sigma(\mathcal{A}_d) \supseteq \mathcal{A}_1 \times \cdots \times \mathcal{A}_d$ and minimality. For the reverse inclusion, let $A_j \in \mathcal{A}_j$, $j = 2, \ldots, d$. Then

$$\sigma(\mathcal{A}_1) \times \{A_2\} \times \cdots \times \{A_d\} \subseteq \sigma\left(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d\right). \tag{†}$$

Indeed, the collection $\mathcal{F}_1$ of all $B_1 \in \sigma(\mathcal{A}_1)$ for which $B_1 \times A_2 \times \cdots \times A_d \in \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d)$ is easily seen to be a σ-field containing $\mathcal{A}_1$ and so by minimality $\mathcal{F}_1 = \sigma(\mathcal{A}_1)$.

Next, let $B_1 \in \sigma(\mathcal{A}_1)$ and $A_j \in \mathcal{A}_j$, $j = 3, \ldots, d$. By (†)

$$\{B_1\} \times \mathcal{A}_2 \times \{A_3\} \times \cdots \times \{A_d\} \subseteq \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d).$$

Arguing as before, this time on the second coordinate, we see that

$$\{B_1\} \times \sigma(\mathcal{A}_2) \times \{A_3\} \cdots \times \{A_d\} \subseteq \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d).$$

We have now shown that

$$\sigma(\mathcal{A}_1) \times \sigma(\mathcal{A}_2) \times \mathcal{A}_3 \cdots \times \mathcal{A}_d \subseteq \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d).$$

Continuing in this manner we eventually obtain the inclusion $\subseteq$ in (1.3). □

1.2.5 Corollary. *Let $d = d_1 + \cdots + d_k$, where $d_j \in \mathbb{N}$. Then*

$$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^{d_1}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}^{d_k}). \tag{1.4}$$

In particular,

$$\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}) \quad (d \text{ factors}).$$

Proof. By definition, $\mathcal{B}(\mathbb{R}^{d_j}) = \sigma(\mathcal{O}_j)$ and $\mathcal{B}(\mathbb{R}^d) = \sigma(\mathcal{O})$, where $\mathcal{O}_j$ is the collection of all open subsets of $\mathbb{R}^{d_j}$ and $\mathcal{O}$ is the collection of all open subsets of $\mathbb{R}^d$. By the theorem,

$$\sigma(\mathcal{O}_1 \times \cdots \times \mathcal{O}_k) = \sigma(\mathcal{O}_1) \times \cdots \times \sigma(\mathcal{O}_k) = \mathcal{B}(\mathbb{R}^{d_1}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}^{d_k}).$$

It therefore suffices to show that

$$\mathcal{O}_1 \times \cdots \times \mathcal{O}_k \subseteq \mathcal{O} \subseteq \mathcal{B}(\mathbb{R}^{d_1}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}^{d_k}); \tag{†}$$

the desired equality (1.4) will then follow by minimality. The first inclusion in (†) follows from the definition of the product topology of $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$ (the latter identified with $\mathbb{R}^d$). For the second inclusion, recall that each $U \in \mathcal{O}$ is a countable union of open intervals $I = (a_1, b_1) \times \cdots \times (a_d, b_d)$. Since each such interval may be written as $I_{d_1} \times \cdots \times I_{d_k}$, where I_{d_j} is a d_j-dimensional open interval, $U \in \mathcal{B}(\mathbb{R}_1^d) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}^{d_k})$. Therefore, (†) holds, completing the proof. □

Pi-Systems and Lambda-Systems

A collection $\mathcal{P}$ of subsets a set X is called a π-**system** if it is closed under finite intersections. Clearly, every field is a π-system, as is the collection of all open (or closed) intervals of $\mathbb{R}$.

A collection $\mathcal{L}$ of subsets a set X is called λ-**system** if it has the following properties:

(a) $X \in \mathcal{L}$.

(b) $A, B \in \mathcal{L}$ and $A \subseteq B \Rightarrow B \setminus A \in \mathcal{L}$. (1.5)

(c) $A_n \in \mathcal{L}$ and $A_n \uparrow A \Rightarrow A \in \mathcal{L}$.

Note that (a) and (b) imply that a λ-system is closed under complements and contains the empty set. The importance of λ-systems is that they provide an indirect method for establishing various properties of certain collections of sets. (See, for example, 1.6.8.) The method is based on Dynkin's π-λ theorem, which makes a connection between π-systems, λ-systems, and σ-fields.

1.2.6 Theorem (Dynkin)**.** *Let $\mathcal{L}$ be a λ-system and $\mathcal{P} \subseteq \mathcal{L}$ a π-system. Then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.*

Proof. Let $\ell(\mathcal{P})$ denote the intersection of all λ-systems containing $\mathcal{P}$. Then $\ell(\mathcal{P})$ is a λ-system, as is easily verified, and $\ell(\mathcal{P}) \subseteq \sigma(\mathcal{P})$. If we show that $\ell(\mathcal{P})$ is a σ-field, it will then follow by minimality that $\sigma(\mathcal{P}) = \ell(\mathcal{P}) \subseteq \mathcal{L}$, establishing the theorem.

To show that $\ell(\mathcal{P})$ is closed under finite intersections, let $A \in \ell(\mathcal{P})$ and define

$$\mathcal{L}_A := \{B \in \ell(\mathcal{P}) : A \cap B \in \ell(\mathcal{P})\}.$$

One easily checks that $\mathcal{L}_A$ is a λ-system. Furthermore, if $A \in \mathcal{P}$, then $\mathcal{P} \subseteq \mathcal{L}_A$, so by minimality $\ell(\mathcal{P}) \subseteq \mathcal{L}_A$. Thus $A \cap B \in \ell(\mathcal{P})$ for all $A \in \mathcal{P}$ and $B \in \ell(\mathcal{P})$. Fixing such a B we have $\mathcal{P} \subseteq \mathcal{L}_B$, hence by minimality $\ell(\mathcal{P}) \subseteq \mathcal{L}_B$. Thus $A,\, B \in \ell(\mathcal{P}) \Rightarrow A \cap B \in \ell(\mathcal{P})$.

Now let (E_n) be a sequence in $\ell(\mathcal{P})$. By the preceding result and induction,

$$A_n := \bigcup_{k=1}^{n} E_k = \left(\bigcap_{k=1}^{n} E_k^c\right)^c \in \ell(\mathcal{P}).$$

By (c) of (1.5), $\bigcup_{k=1}^\infty E_k = \bigcup_{n=1}^\infty A_n \in \ell(\mathcal{P})$. Therefore, $\ell(\mathcal{P})$ is a σ-field, completing the proof. $\square$

Exercises

1.1 Let A, B, C, A_n, $B_n \subseteq X$. Verify the following:

(a) $\mathbf{1}_{A\Delta B} = |\mathbf{1}_A - \mathbf{1}_B|$.

(b) $\big(A \triangle B\big)^c = A^c \triangle B = A \triangle B^c$.

(c) $A^c \triangle B^c = A \triangle B$.

(d) $\big(A \triangle B\big) \cap C = (A \cap C) \triangle (B \cap C)$.

(e) $\left(\bigcup_{n=1}^\infty A_n\right) \triangle \left(\bigcup_{n=1}^\infty B_n\right) \subseteq \bigcup_{n=1}^\infty A_n \triangle B_n$.

1.2 Let A_n, $B_n \subseteq X$. Verify the following:

(a) $x \in \underline{\lim}_n A_n$ iff $x \in A_n$ for all sufficiently large n.

(b) $x \in \overline{\lim}_n A_n$ iff $x \in A_n$ for infinitely many n.

(c) $\underline{\lim}_n A_n \subseteq \overline{\lim}_n A_n$.

(d) $\big(\underline{\lim}_n A_n\big)^c = \overline{\lim}_n A_n^c$.

(e) $\big(\overline{\lim}_n A_n\big)^c = \underline{\lim}_n A_n^c$.

(f) $\overline{\lim}_n (A_n \cap B_n) \subseteq \overline{\lim}_n A_n \cap \overline{\lim}_n B_n$.

(g) $\overline{\lim}_n (A_n \cup B_n) = \overline{\lim}_n A_n \cup \overline{\lim}_n B_n$.

(h) $\underline{\lim}_n (A_n \cap B_n) = \underline{\lim}_n A_n \cap \underline{\lim}_n B_n$.

(i) $\underline{\lim}_n (A_n \cup B_n) \supseteq \underline{\lim}_n A_n \cup \underline{\lim}_n B_n$.

Show that the inclusions in (c), (f), and (i) may be strict.

1.3 For $A_n \subseteq X$, write $A_n \to A$ if $\overline{\lim}_n A_n = \underline{\lim}_n A_n = A$. Let $A_n \to A$ and $B_n \to B$. Show that
(a) $A_n \cup B_n \to A \cup B$. (b) $A_n \cap B_n \to A \cap B$. (c) $A_n^c \to A^c$. (d) $A_n \triangle B_n \to A \triangle B$.

1.4 Let $A_n, A \subseteq X$ and set $B = \underline{\lim}_n A_n$ and $C = \overline{\lim}_n A_n$. Prove that
(a) $\mathbf{1}_B = \underline{\lim}_n \mathbf{1}_{A_n}$. (b) $\mathbf{1}_C = \overline{\lim}_n \mathbf{1}_{A_n}$ (c) $A_n \to A$ iff $\mathbf{1}_{A_n} \to \mathbf{1}_A$.

1.5 Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and set $A_n = (-\infty, a_n)$ and $B_n = (a_n, \infty)$. Prove:

(a) $x \in \overline{\lim}_n A_n \Rightarrow x \leq \overline{\lim}_n a_n$.
(b) $x < \overline{\lim}_n a_n \Rightarrow x \in \overline{\lim}_n A_n$.
(c) $x \in \underline{\lim}_n A_n \Rightarrow x \leq \underline{\lim}_n a_n$.
(d) $x < \underline{\lim}_n a_n \Rightarrow x \in \underline{\lim}_n A_n$.
(e) $x \in \overline{\lim}_n B_n \Rightarrow \underline{\lim}_n a_n \leq x$.

1.6 Determine all sets in the field on $X = \{1, 2, 3, 4, 5, 6\}$ generated by the sets

(a) $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$.
(b) $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$.
(c) $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{3, 4, 5, 6\}$.

1.7 Let $\mathcal{F}$ be a σ-field on X and $E \subseteq X$. Show that $\sigma\big(\mathcal{F} \cup \{E\}\big)$ consists of all sets of the form $(A \cap E) \cup (B \cap E^c)$, A, $B \in \mathcal{F}$.

1.8 Let $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $X \in \mathcal{F}$ and $A \setminus B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

1.9 Show that if X is infinite, then the field consisting of all finite or cofinite sets is not a σ-field.

1.10 Let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a sequence of σ-fields on X such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$. Show that $\mathcal{F} := \bigcup_{n=1}^\infty \mathcal{F}_n$ is a field. Show by example that $\mathcal{F}$ need not be a σ-field.

1.11 Find examples of fields $\mathcal{F}$ and $\mathcal{G}$ on $X = \{1, 2, 3\}$ such that $\mathcal{F} \cup \mathcal{G}$ is not a field.

1.12 Describe the σ-field $\mathcal{F}$ on $(0, 1)$ generated by all singletons $\{x\}$, $x \in (0, 1)$. Show that $\mathcal{F}$ is contained in $\mathcal{B}(0, 1)$ and contains no proper open subinterval of $(0, 1)$.

1.13 Let $\mathcal{F}$ be the collection of all finite disjoint unions of intervals $[a, b) \subseteq [0, 1)$. Show that $\mathcal{F}$ is a field on $[0, 1)$ but not a σ-field.

1.14 Let $\mathcal{A} \subseteq \mathcal{P}(X)$. Show that $\sigma\big(\varphi(\mathcal{A})\big) = \sigma(\mathcal{A})$.

1.15 Let $\mathcal{F}_f$ denote the field consisting of the subsets of X that are either finite or cofinite. Show that $\sigma(\mathcal{F}_f)$ is the σ-field $\mathcal{F}_c$ consisting of the countable or cocountable subsets of X.

1.16 Show that $\mathcal{B}(\mathbb{R}^d)$ is generated by the collection

(a) $\mathcal{K}$ of all compact sets.
(b) $\mathcal{I}_r$ of all intervals $(a_1, \infty) \times \cdots \times (a_d, \infty)$, $a_j \in \mathbb{Q}$.

1.17 Let $\mathcal{F}$ be a field. Prove that the following are equivalent:

(a) $\mathcal{F}$ is a σ-field.
(b) $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$ for every sequence of disjoint sets $A_n \in \mathcal{F}$.
(c) $\bigcup_{n=1}^\infty B_n \in \mathcal{F}$ for every increasing sequence of sets $B_n \in \mathcal{F}$.

1.18 Let $\mathcal{A} \subseteq \mathcal{P}(X)$ and $E \subseteq X$. Prove that $\sigma(\mathcal{A} \cap E) = \sigma(\mathcal{A}) \cap E$.

1.19 Let X be a topological space and let $E \subseteq X$ have the relative topology. Prove that $\mathcal{B}(X) \cap E = \mathcal{B}(E)$.

1.20 [$\downarrow$2.30] Let a, $b \in \mathbb{R}$ and let $[a, b]$ and (a, b) have the relative topology from $\mathbb{R}$. Show that $\mathcal{B}([a, b])$ consists of the sets B, $B \cup \{a\}$, $B \cup \{b\}$, and $B \cup \{a, b\}$ where $B \in \mathcal{B}\big((a, b)\big)$.

1.21 For $j = 1, \dots, d$, let $\mathcal{A}_j \subseteq \mathcal{P}(X_j)$ and $E_j \in \mathcal{P}(X_j)$. Set $E := E_1 \times \cdots \times E_d$. Show that $\sigma(\mathcal{A}_1 \cap E_1) \otimes \cdots \otimes \sigma(\mathcal{A}_d \cap E_d) = \sigma(\mathcal{A}_1 \times \cdots \times \mathcal{A}_d) \cap E$.

1.22 Let $B \in \mathcal{B}(\mathbb{R}^d)$, $\boldsymbol{x} \in \mathbb{R}^d$, and $r \in \mathbb{R}$. Prove that $B + \boldsymbol{x} := \{\boldsymbol{b} + \boldsymbol{x} : \boldsymbol{b} \in B\}$ and $rB := \{r\boldsymbol{b} : \boldsymbol{b} \in B\}$ are Borel sets.

1.23 Let $\mathcal{A} \subseteq \mathcal{P}(X)$ and let $\mathcal{F}$ be the union of all σ-fields $\sigma(\mathcal{C})$, where $\mathcal{C}$ is a countable subfamily of $\mathcal{A}$. Prove that $\mathcal{F} = \sigma(\mathcal{A})$.

1.24 Let $\mathcal{F} = \{B_1, \ldots, B_m\}$ be a finite field on X. Show that there exists a finite partition $\mathcal{A}$ of X by sets in $\mathcal{F}$ such that every member of $\mathcal{F}$ is a union of members of $\mathcal{A}$. ⟦Consider $C_1 \cap \cdots \cap C_m$, where $C_j = B_j$ or B_j^c. ⟧

1.25 Show that every infinite σ-field $\mathcal{F}$ has an infinite sequence of disjoint nonempty sets. Conclude that $\mathcal{F}$ has cardinality at least that of the continuum. Conclude that no σ-field can have cardinality $\aleph_0$. Find a *field* that has cardinality $\aleph_0$.

1.26 A nonempty collection $\mathcal{M}$ of subsets of X is a **monotone class** if for any sequence $\{A_n\}$ in $\mathcal{M}$, $A_n \uparrow A$ or $A_n \downarrow A \Rightarrow A \in \mathcal{M}$. Carry out steps (a)–(f) below to prove the **monotone class theorem**, due to Halmos: *If $\mathcal{F}$ is a field, $\mathcal{M}$ is a monotone class, and $\mathcal{F} \subseteq \mathcal{M}$, then $\sigma(\mathcal{F}) \subseteq \mathcal{M}$.*

(a) Show that a monotone class that is closed under finite unions (intersections) is closed under countable unions (intersections).

(b) Let $m(\mathcal{F})$ denote the intersection of all monotone classes containing $\mathcal{F}$. Show that $m(\mathcal{F})$ is a monotone class.

(c) Show that $\mathcal{A} := \{A \in m(\mathcal{F}) : A^c \in m(\mathcal{F})\}$ is monotone and $m(\mathcal{F}) = \mathcal{A}$. Conclude that $m(\mathcal{F})$ is closed under complements.

(d) Let $\mathcal{B} = \{B \in m(\mathcal{F}) : A \cup B \in m(\mathcal{F}) \text{ for all } A \in \mathcal{F}\}$. Show that $\mathcal{B}$ is a monotone class and $\mathcal{B} = m(\mathcal{F})$. Conclude that $A \cup B \in m(\mathcal{F})$ for all $B \in m(\mathcal{F})$ and all $A \in \mathcal{F}$.

(e) Let $\mathcal{C} = \{C \in m(\mathcal{F}) : C \cup B \in m(\mathcal{F}) \text{ for all } B \in m(\mathcal{F})\}$. Show that $\mathcal{C}$ is monotone and $\mathcal{C} = m(\mathcal{F})$. Conclude that $m(\mathcal{F})$ is closed under finite unions.

(f) Show that $m(\mathcal{F})$ is closed under countable unions. Conclude that $\sigma(\mathcal{F}) \subseteq m(\mathcal{F}) \subseteq \mathcal{M}$.

1.3 Measures

Set Functions

Let X be a nonempty set. A collection of subsets of X containing the empty set is called a **paving of** X. A function μ on a paving $\mathcal{A}$ of X that takes values in $\overline{\mathbb{R}}$ is called a **set function on** $\mathcal{A}$. Until Chapter 5, we consider only **nonnegative set functions**, that is, those taking values in $[0, \infty]$. An important example is the function that assigns the length $b - a$ to intervals $[a, b]$. This set function and its d-dimensional generalization will be examined in detail in §1.7.

Let μ be a nonnegative set function on a paving $\mathcal{A}$ and let $A_1, A_2, \ldots \in \mathcal{A}$. Then μ is said to be

- **monotone** if $A_1 \subseteq A_2$ implies $\mu(A_1) \le \mu(A_2)$.
- **finitely additive** if $A := \bigcup_{k=1}^n A_k$ disjoint and $A \in \mathcal{A}$ implies $\mu(A) = \sum_{k=1}^n \mu(A_k)$.
- **finitely subadditive** if $A := \bigcup_{k=1}^n A_k \in \mathcal{A}$ implies $\mu(A) \le \sum_{k=1}^n \mu(A_k)$.
- **countably additive** if $A := \bigcup_{n=1}^\infty A_n$ disjoint and $A \in \mathcal{A}$ implies $\mu(A) = \sum_{n=1}^\infty \mu(A_n)$.
- **countably subadditive** if $A := \bigcup_{n=1}^\infty A_n \in \mathcal{A}$ implies $\mu(A) \le \sum_{n=1}^\infty \mu(A_n)$.
- **finite** if $\mu(A) < \infty$ for every $A \in \mathcal{A}$.
- **σ-finite** if there exist pairwise disjoint $X_1, X_2, \ldots \in \mathcal{A}$ with union X and $\mu(X_n) < \infty$.
- a **measure on** $\mathcal{A}$ if μ is countably additive and $\mu(\emptyset) = 0$.

If μ is a measure on a σ-field $\mathcal{F}$, then the triple $(X, \mathcal{F}, \mu)$ is called a **measure space**. A member E of $\mathcal{F}$ that is a countable union of sets of finite measure is called a **σ-finite set**. If $\mu(X) = 1$, then μ is said to be a **probability measure**. Note that a measure on a field is finitely additive: simply apply countable additivity to the sequence $A_1, \ldots, A_n, \emptyset, \emptyset, \ldots$.

Notation. In the sequel, if μ is a set function defined on intervals we write $\mu(a, b)$ for $\mu((a, b))$, $\mu[a, b]$ for $\mu([a, b])$, etc. No confusion should arise from these abbreviations, as context will make clear the intended meaning.

Properties and Examples of Measures

1.3.1 Proposition. *A measure μ on a σ-field $\mathcal{F}$ is monotone and countably subadditive. Moreover, for $A_n \in \mathcal{F}$ the following hold:*

- **(a)** (*Continuity at A from below*). *$A_n \uparrow A$ implies $\mu(A_n) \uparrow \mu(A)$.*
- **(b)** (*Continuity at A from above*). *$A_n \downarrow A$ and $\mu(A_1) < \infty$ implies $\mu(A_n) \downarrow \mu(A)$.*

Proof. If $A_1 \subseteq A_2$ then $\mu(A_2) = \mu(A_2 \setminus A_1) + \mu(A_1) \ge \mu(A_1)$, hence μ is monotone. For subadditivity use (1.2), countable additivity, and monotonicity:

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \mu(A_1) + \mu(A_2 \cap A_1^c) + \mu(A_3 \cap A_1^c \cap A_2^c) + \cdots \le \sum_{k=1}^{\infty} \mu(A_k).$$

Part (a) is clear if some A_k has infinite measure, so assume $\mu(A_k) < \infty$ for all k. Set $A_0 = \emptyset$ and $E_k = A_k \setminus A_{k-1}$. Then A is the disjoint union $\bigcup_{k=1}^{\infty} E_k$, hence

$$\mu(A) = \sum_{k=1}^{\infty} \mu(E_k) = \lim_n \sum_{k=1}^{n} \big[\mu(A_k) - \mu(A_{k-1})\big] = \lim_n \mu(A_n).$$

For (b), note that $A_1 \setminus A_n \uparrow A_1 \setminus A$, hence, by (a),

$$\mu(A_1) - \mu(A) = \mu(A_1 \setminus A) = \lim_n \mu(A_1 \setminus A_n) = \mu(A_1) - \lim_n \mu(A_n). \qquad \square$$

The preceding proposition has a converse:

1.3.2 Proposition. *Let μ be a finitely additive, nonnegative set function on a field $\mathcal{F}$.*

- **(a)** *If μ is continuous from below, then μ is a measure.*
- **(b)** *If $\mu(X) < \infty$ and μ is continuous at $\emptyset$ from above, then μ is a measure.*

Proof. For (a), let $\{A_n\}$ be a sequence of disjoint sets in $\mathcal{F}$ with union $A \in \mathcal{F}$ and set $B_n := \bigcup_{k=1}^{n} A_k$. Then $B_n \in \mathcal{F}$ and $B_n \uparrow A$. By finite additivity and continuity from below,

$$\sum_{k=1}^{\infty} \mu(A_k) = \lim_n \sum_{k=1}^{n} \mu(A_k) = \lim_n \mu(B_n) = \mu(A).$$

The proof of (b) is left as an exercise (1.39). $\square$

1.3.3 Examples.

(a) Set $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ if $A \ne \emptyset$. Then μ is a measure on $\mathcal{P}(X)$.

(b) Let X be an infinite set and define $\mu(A) = 0$ if A is countable and $\mu(A) = \infty$ otherwise. Then μ is a measure on $\mathcal{P}(X)$.

(c) Let X be uncountable and $\mathcal{F}$ the σ-field of countable or cocountable subsets of X (see 1.2.1(c)). Define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is cocountable. Then μ is a probability measure on $\mathcal{F}$.

(d) *Dirac measure.* Let $(X, \mathcal{F})$ be a measurable space. For $x \in X$ and $A \in \mathcal{F}$ define $\delta_x(A) = \mathbf{1}_A(x)$. Then δ_x is a probability measure on $\mathcal{F}$.

(e) If μ_j are measures on a σ-field $\mathcal{F}$ and $a_j \geq 0$, then $\sum_{j=1}^{n} a_j \mu_j$ is a measure on $\mathcal{F}$. In particular, a nonnegative linear combination of Dirac measures is a measure.

(f) If $(X, \mathcal{F}, \mu)$ is a measure space and $E \in \mathcal{F}$, then $\mu_E(A) := \mu(A \cap E)$ defines a measure on $\mathcal{F}$. Note that μ_E agrees with μ on the trace $\mathcal{F} \cap E$.

(g) *Counting measure.* Let X be a nonempty set. For $A \subseteq X$ let $\mu(A)$ be the number of elements in A if A is finite and $\mu(A) = \infty$ otherwise. Then μ is clearly finitely additive on $\mathcal{P}(X)$. To show that μ is a measure, let $A_n \uparrow A$. If there exists an m such that $A_m = A$, then $A_n = A$ for all $n \geq m$ and so, trivially, $\mu(A_n) \uparrow \mu(A)$. On the other hand, if no such m exists, then A must be infinite and $A_{n_{k-1}} \subsetneqq A_{n_k}$ for some sequence of indices. Since $\mu(A_{n_k}) \geq \mu(A_{n_{k-1}}) + 1$,

$$\lim_n \mu(A_n) = \lim_k \mu(A_{n_k}) = \infty = \mu(A).$$

By 1.3.2, μ measure on $\mathcal{P}(X)$.

(h) *Infinite series measure.* For an arbitrary sequence (p_n) in $[0, \infty)$, define

$$\mu(E) = \sum_{k \in E} p_k, \quad E \subseteq \mathbb{N},$$

where the sum may be infinite. (By convention, the sum over the empty set is zero.) The rearrangement theorem for nonnegative series implies that μ is well-defined and finitely additive. Let $A_n \uparrow A$. If A is finite, then eventually $A_n = A$, so obviously $\mu(A_n) \uparrow \mu(A)$. If A is infinite, then $\mu(A)$ may be written as an infinite series $\mu(A) = \sum_{k=1}^{\infty} p_{n_k}$. Let $r < \mu(A)$, choose k such that $\sum_{i=1}^{k} p_{n_i} > r$, and choose m so that A_m contains the indices $n_1, \ldots, n_k$. Then $\mu(A_n) \geq \mu(A_m) > r$ for all $n \geq m$. Since r was arbitrary, $\mu(A_n) \to \mu(A)$. By 1.3.2, μ is a measure on $\mathcal{P}(\mathbb{N})$. Note that if $p_k \equiv 1$, then μ is simply counting measure on $\mathbb{N}$. ◊

Exercises

1.27 Let $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\emptyset \in \mathcal{A}$. Show that if μ is a countably additive, finite set function on $\mathcal{A}$, then $\mu(\emptyset) = 0$.

1.28 Verify that the set functions defined in 1.3.3 (c) and (d) are measures.

1.29 Give an example of a measure μ on a σ-field $\mathcal{F}$ and a sequence of sets $A_n \in \mathcal{F}$ decreasing to A such that $\lim_n \mu(A_n) \neq \mu(A)$.

1.30 [$\uparrow$ 1.2.1] Let $\mathcal{F}$ be the field of finite or cofinite subsets of X and define $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A is cofinite. (a) Show that μ is finitely additive but in general is not countably additive. (b) Show that μ is countably additive if X is uncountable.

1.31 Let μ be a finitely additive, nonnegative set function on a field $\mathcal{F}$. Prove that if $\mu(A)$ and $\mu(B)$ are finite, then $|\mu(A) - \mu(B)| \leq \mu(A \,\triangle\, B)$.

1.32 (Inclusion-exclusion I). Let μ be a finitely additive nonnegative set function on a field $\mathcal{F}$. Prove that $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$.

1.33 Let μ be a finitely additive, nonnegative set function on a field $\mathcal{F}$ and let $A, B \in \mathcal{F}$ with $\mu(B) = 0$. Show that $\mu(A \cup B) = \mu(A \setminus B) = \mu(A)$.

1.34 (Inclusion-exclusion II). Let μ be a finitely additive, nonnegative set function on a field $\mathcal{F}$ and let $A_1, \ldots, A_n \in \mathcal{F}$ with union A such that $\mu(A) < \infty$. Prove that for $n \geq 2$

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i) - \sum_{1\leq i<j\leq n}^{n} \mu(A_i \cap A_j) + \sum_{1\leq i<j<k\leq n}^{n} \mu(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1}\mu(A_1 \cap \cdots \cap A_n).$$

1.35 (Inclusion-exclusion III). Let μ be a finitely additive, nonnegative set function on a field $\mathcal{F}$ with $\mu(X) < \infty$ and let $B_1, \ldots, B_n \in \mathcal{F}$ with intersection B. Prove that for $n \geq 2$,

$$\mu(B) = \sum_{i=1}^{n} \mu(B_i) - \sum_{1\leq i<j\leq n}^{n} \mu(B_i \cup B_j) + \sum_{1\leq i<j<k\leq n}^{n} \mu(B_i \cup B_j \cup B_k) - \cdots + (-1)^{n-1}\mu(B_1 \cup \cdots \cup B_n).$$

1.36 Let $(X, \mathcal{F}, \mu)$ be a measure space and let $A_n \in \mathcal{F}$ such that $\mu(A_m \cap A_n) = 0$ for $m \neq n$. Prove that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

1.37 [$\downarrow$5.3.2] Let $(X, \mathcal{F}, \mu)$ be a measure space and $A_n \in \mathcal{F}$. Prove:

(a) $\mu\left(\underline{\lim}_n A_n\right) \leq \underline{\lim}_n \mu(A_n)$.

(b) $\mu\left(\overline{\lim}_n A_n\right) \geq \overline{\lim}_n \mu(A_n)$ if $\mu\left(\bigcup_n A_n\right) < \infty$.

(c) $\mu\left(\overline{\lim}_n A_n\right) = 0$ if $\sum_n \mu(A_n) < \infty$.

1.38 Let $(X, \mathcal{F})$ be a measurable space and let $x_1, x_2 \in X$. For $A \in \mathcal{P}(X)$, define $\mu(A) = 1$ if $\{x_1, x_2\} \subseteq A$ and $\mu(A) = 0$ otherwise. Prove that μ is continuous from below. Is μ a measure?

1.39 Prove 1.3.2(b).

1.40 [$\downarrow$Ex. 3.3] Let μ_n be a sequence of measures on a σ-field $\mathcal{F}$ on X such that $\mu_n(A) \leq \mu_{n+1}(A)$ for all $A \in \mathcal{F}$. Define the set function μ on $\mathcal{F}$ by $\mu(A) = \lim_n \mu_n(A)$. Prove that μ is a measure.

1.41 Let μ_n be a sequence of measures on a σ-field $\mathcal{F}$ on X and define μ on $\mathcal{F}$ by $\mu(A) = \sum_n \mu_n(A)$. Prove that μ is a measure.

1.42 Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Show that there can be at most countably many pairwise disjoint sets of positive measure.

1.43 Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and $\mathcal{E}$ a collection of pairwise disjoint members of $\mathcal{F}$. Show that for any $A \in \mathcal{F}$, $\mu(A \cap E) > 0$ for at most countably many members of $\mathcal{E}$.

1.44 Let $(X, \mathcal{F}, \mu)$ be a measure space and for $A \in \mathcal{F}$ define

$$\mu_0(A) = \sup\{\mu(B) : B \in \mathcal{F},\ B \subseteq A \text{ and } \mu(B) < \infty\}.$$

Show that μ_0 is a measure on $\mathcal{F}$. Show also that $\mu_0 = \mu$ iff the following condition holds: For each $A \in \mathcal{F}$ with $\mu(A) = \infty$ there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$.

1.45 Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{E_k\}$ be a sequence in $\mathcal{F}$. For fixed $m \in \mathbb{N}$, let A denote the set of all x such that $x \in E_k$ for exactly m values of k; B the set of all x such that $x \in E_k$ for finitely many and at least m values of k; and C the set of all x such that $x \in E_k$ for at most m values of k. Prove that A, B, $C \in \mathcal{F}$. If $s(D) := \sum_{k=1}^{\infty} \mu(D \cap E_k)$, prove that

(a) $\mu(A) = s(A)/m$. (b) $\mu(B) \geq s(B)/m$. (c) $\mu(C) \leq s(C)/m$.

1.4 Complete Measure Spaces

A measure space $(X, \mathcal{F}, \mu)$ is said to be **complete** if

$$M \in \mathcal{F},\ \mu(M) = 0,\ \text{and } N \subseteq M \Rightarrow N \in \mathcal{F}.$$

Examples (a)–(c), (g), and (h) of 1.3.3 are complete measure spaces. In this section we show that any measure space $(X, \mathcal{F}, \mu)$ may be enlarged in a minimal way to produce a complete measure space. The following simple example illustrates the basic idea behind the construction.

1.4.1 Example. Let $X = \{1, 2, 3\}$ and $\mathcal{F} = \big\{\emptyset, \{1\}, \{2, 3\}, X\big\}$. The measure μ defined by $\mu\{1\} = 1$ and $\mu\{2, 3\} = 0$ is not complete. However, by enlarging $\mathcal{F}$ to include $\{2\}$, $\{3\}$ and defining a new measure $\overline{\mu}$ on the augmented σ-field so that $\overline{\mu}\{1\} = 1$ and $\overline{\mu}\{2\} = \overline{\mu}\{3\} = 0$, we obtain an extension of $(X, \mathcal{F}, \mu)$ that is complete. $\Diamond$

Completion Theorem

Here is the general technique for completing a measure space. Part (a) of the theorem gives the construction and part (b) describes a minimality property of a completion.

1.4.2 Theorem. *Let $(X, \mathcal{F}, \mu)$ be a measure space. Define*

$$\mathcal{F}_\mu := \big\{A \cup N : A \in \mathcal{F},\ N \subseteq M \in \mathcal{F},\ \mu(M) = 0\big\} \quad \textit{and} \quad \overline{\mu}(A \cup N) := \mu(A). \tag{1.6}$$

(a) *$\mathcal{F}_\mu$ is a σ-field containing $\mathcal{F}$ and $\overline{\mu}$ is a measure on $\mathcal{F}_\mu$ that extends μ such that $(X, \mathcal{F}_\mu, \overline{\mu})$ is complete.*

(b) *If $(X, \mathcal{G}, \nu)$ is a complete measure space such that $\mathcal{F} \subseteq \mathcal{G}$ and ν is an extension of μ, then $\mathcal{F}_\mu \subseteq \mathcal{G}$ and the restriction of ν to $\mathcal{F}_\mu$ is $\overline{\mu}$.*

Proof. (a) To see that $\overline{\mu}$ is well-defined, let $A_1 \cup N_1 = A_2 \cup N_2$, where $N_j \subseteq M_j$, A_j, $M_j \in \mathcal{F}$ and $\mu(M_j) = 0$. Then $A_1 \subseteq A_2 \cup M_2$ and $A_2 \subseteq A_1 \cup M_1$, hence $\mu(A_2) = \mu(A_1)$.

Clearly, $\mathcal{F} \subseteq \mathcal{F}_\mu$. To see that $\mathcal{F}_\mu$ is closed under complements note that in the notation of (1.6)

$$(A \cup N)^c = (A^c \cap M^c) \cup (A^c \cap N^c \cap M), \quad A^c \cap M^c \in \mathcal{F} \text{ and } A^c \cap N^c \cap M \subseteq M.$$

For closure under countable unions, let $B_n := A_n \cup N_n \in \mathcal{F}_\mu$ and $B := \bigcup_n B_n$, where $N_n \subseteq M_n$, A_n, $M_n \in \mathcal{F}$, and $\mu(M_n) = 0$. Then

$$B = A \cup N, \quad \text{where } A := \bigcup_{n=1}^\infty A_n \text{ and } N := \bigcup_{n=1}^\infty N_n \subseteq M := \bigcup_{n=1}^\infty M_n.$$

Since $\mu(M) = 0$, $B \in \mathcal{F}_\mu$. Moreover, if the sets B_n are disjoint, then

$$\overline{\mu}(B) = \mu(A) = \sum_n \mu(A_n) = \sum_n \overline{\mu}(B_n).$$

Therefore, $\mathcal{F}_\mu$ is a σ-field and $\overline{\mu}$ is a measure on $\mathcal{F}_\mu$. Clearly, $(X, \mathcal{F}_\mu, \overline{\mu})$ is complete.

(b) Let A, N and M be as in (1.6). Then $\nu(M) = \mu(M) = 0$, hence, since $(X, \mathcal{G}, \nu)$ is complete, $N \in \mathcal{G}$. Therefore, $A \cup N \in \mathcal{G}$ and $\overline{\mu}(A \cup N) = \mu(A) = \nu(A) = \nu(A \cup N)$, so ν is an extension of $\overline{\mu}$. $\square$

Note that the completion theorem produces nothing new if $(X, \mathcal{F}, \mu)$ is already complete, since then the sets N in the above construction are already in $\mathcal{F}$.

Null Sets

The sets N in the completion theorem, namely the subsets of $\mathcal{F}$-measurable sets M with measure zero, are called **μ-null sets**. Such sets appear throughout measure theory, frequently in the following context:

A property $P(x)$ of points $x \in X$ is said to hold **μ-almost everywhere**, abbreviated **μ-a.e.**, if the set of all x for which $P(x)$ is false is a μ-null set, that is,

$$\overline{\mu}\,\{x \in X : P(x) \text{ is false}\} = 0.$$

In this case we also say that the property $P(x)$ holds for **μ-almost all** x, abbreviated **μ-a.a.** x. If the measure is clear from context we drop the qualifier μ and simply write a.e. or a.a. For example, if a function f in 1.4.1 is defined by $f(j) = j$, then $f = 1$ a.e. For an example with far reaching implications, consider functions $f_n,\ f : X \to \mathbb{C}$. The notation $f_n \to f$ a.e. then means that

$$\overline{\mu}\{x \in X : \lim_n f_n(x) \neq f(x)\} = 0.$$

This type of convergence will be examined in Chapter 2.

Exercises

1.46 [$\uparrow$1.3.3(d).] Let $(X, \mathcal{F})$ be a measurable space, E a finite subset of X, and $\mu := \sum_{x\in E} \delta_x$. Describe the completion of $(X, \mathcal{F}, \mu)$.

1.47 Show that if $\mathcal{G} \subseteq \mathcal{F}$ are sigma fields, μ is a measure on $\mathcal{F}$, and $\nu = \mu\big|_{\mathcal{G}}$, then $\mathcal{G}_\nu \subseteq \mathcal{F}_\mu$ and $\overline{\nu} = \overline{\mu}\big|_{\mathcal{G}_\nu}$.

1.48 [$\uparrow$1.44] Prove that $\overline{\mu_0} = \overline{\mu}_0$.

1.49 Let $\{\mathcal{F}^i : i \in \mathfrak{I}\}$ be a collection of σ-fields on X and μ a measure on $\mathcal{G} := \sigma\big(\bigcup_i \mathcal{F}^i\big)$. For each i let μ_i denote the restriction of μ to $\mathcal{F}^i$. Show that $\mathcal{G}_\mu = \mathcal{H}_\mu$, where $\mathcal{H} := \sigma\big(\bigcup_i \mathcal{F}^i_{\mu_i}\big)$.

1.50 Let ν and η be measures on a σ-field $\mathcal{F}$ and set $\mu := \nu + \eta$. Show that $\mathcal{F}_\mu \subseteq \mathcal{F}_\nu \cap \mathcal{F}_\eta$ and $\overline{\mu} := \overline{\nu} + \overline{\eta}$ on $\mathcal{F}_\mu$.

1.51 [$\uparrow$1.3.3(f)] Let $E \in \mathcal{F}$. Prove that $\mathcal{F}_{\mu_E} \cap E = \mathcal{F}_\mu \cap E$ and $\overline{\mu_E} = \overline{\mu}_E$ on $\mathcal{F}_{\mu_E}$.

1.52 Let $(X, \mathcal{F}, \mu)$ be a finite measure space. For $E \subseteq X$ define

$$\mu_*(E) = \sup\{\mu(A) : A \in \mathcal{F},\ A \subseteq E\} \text{ and } \mu^*(E) = \inf\{\mu(B) : B \in \mathcal{F},\ B \supseteq E\}.$$

Show that $\mathcal{F}_\mu = \{E \subseteq X : \mu_*(E) = \mu^*(E)\}$.

1.5 Outer Measure and Measurability

As mentioned in the introduction to the chapter, the construction of a measure generally begins with a collection $\mathcal{A}$ of "elementary" subsets of X and a set function μ on $\mathcal{A}$, and culminates with an extension of μ to a measure on a σ-field containing $\mathcal{A}$. Of course, there may be several σ fields containing $\mathcal{A}$, $\mathcal{P}(X)$ being an obvious one. However, in many cases it is impossible to extend μ to $\mathcal{P}(X)$. For example, in §1.7 it is shown that the length set-function on the collection bounded intervals of $\mathbb{R}$ cannot be extended to a measure on $\mathcal{P}(\mathbb{R})$. In general, the best one can hope for is an extension of μ to the completion of the σ-field generated by $\mathcal{A}$. This is accomplished by first constructing a related set function on $\mathcal{P}(X)$, called *outer measure*, and then restricting this function to the class of so-called *measurable sets*. The details follow.

Construction of an Outer Measure

An **outer measure** on a nonempty set X is a nonnegative, monotone, countably subadditive set function μ^* on $\mathcal{P}(X)$ such that $\mu^*(\emptyset) = 0$. Clearly, every measure on $\mathcal{P}(X)$ is an outer measure. In particular, the set function that assigns 0 to the empty set and ∞ to every nonempty set is an outer measure. By contrast, the set function that assigns 0 to the empty set and 1 to every nonempty set is an outer measure that is not a measure.

The following proposition describes a general class of outer measures which are typically not measures. The outer measure μ^* defined in (1.7) is said to be **generated by the pair** $(\mathcal{A}, \mu)$. The sequences (A_n) in (1.7) are said to **cover** E.

1.5.1 Proposition. *Let $\mathcal{A}$ be a paving of X and let μ be a nonnegative set function on $\mathcal{A}$ such that $\mu(\emptyset) = 0$. Define a set function μ^* on $\mathcal{P}(X)$ by*

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \ A_n \in \mathcal{A} \text{ and } E \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, \tag{1.7}$$

where $\inf \emptyset := \infty$. *Then μ^* is an outer measure.*

Proof. That $\mu^*(\emptyset) = 0$ can be seen by taking as a cover the sequence $A_1 = A_2 = \cdots = \emptyset$. Monotonicity of μ^* follows from the observation that if $A \subseteq B$, then every cover of B is a cover of A. For countable subadditivity, let $E_n \in \mathcal{P}(X)$ and $E := \bigcup_{n=1}^{\infty} E_n$. We may assume that $\sum_{n=1}^{\infty} \mu^*(E_n) < \infty$. Given $\varepsilon > 0$, for each n choose a cover $\{A_{n,j}\}_j$ of E_n in $\mathcal{A}$ such that

$$\sum_{j=1}^{\infty} \mu(A_{n,j}) < \mu^*(E_n) + \varepsilon/2^n.$$

Since $\{A_{n,j}\}_{n,j}$ is a cover of E,

$$\mu^*(E) \le \sum_{n,j} \mu(A_{n,j}) < \sum_n \mu^*(E_n) + \varepsilon.$$

Thus $\mu^*(E) \le \sum_n \mu^*(E_n)$, as required. $\square$

Carathéodory's Theorem

Let μ^* be any outer measure on X. A subset E of X is said to be **μ^*-measurable** if

$$\mu^*(C) = \mu^*(C \cap E) + \mu^*(C \cap E^c) \quad \text{for all } C \subseteq X. \tag{1.8}$$

The definition asserts that E "splits" the outer measure of each subset C of X, a property that may be seen as a precursor to finite additivity. Note that by subadditivity the inequality $\le$ in (1.8) always holds. Thus the measurability criterion singles out precisely those sets E for which the inequality $\ge$ in (1.8) is satisfied. The collection of all μ^*-measurable subsets of X is denoted by $\mathcal{M}(\mu^*)$. Here is the main result regarding outer measure.

1.5.2 Theorem (Carathéodory)**.** *Let μ^* be an outer measure on X. Then $\mathcal{M} := \mathcal{M}(\mu^*)$ is a σ-field and the restriction $\overline{\mu} := \mu^*\big|_{\mathcal{M}}$ is a complete measure.*

Proof. Clearly, $\emptyset$, $X \in \mathcal{M}$, and since E and E^c appear symmetrically in (1.8), $E^c \in \mathcal{M}$ iff $E \in \mathcal{M}$. Furthermore, if $\mu^*(E) = 0$, then, by monotonicity,

$$\mu^*(C \cap E) + \mu^*(C \cap E^c) \le \mu^*(E) + \mu^*(C \cap E^c) = \mu^*(C \cap E^c) \le \mu^*(C),$$

hence $E \in \mathcal{M}$. Thus $\mathcal{M}$ contains all sets of outer measure zero.

It remains to show that for any sequence (E_n) in $\mathcal{M}$,

$$\text{(a)} \quad \bigcup_{n=1}^{\infty} E_n \in \mathcal{M} \quad \text{and} \quad \text{(b)} \quad \mu^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu^*(E_n) \quad \text{if the union is disjoint.}$$

The verifications of (a) and (b) are carried out in the following steps. For convenience, call a set C for which the equality in (1.8) holds a **test set for** E.

(1) *$\mathcal{M}$ closed under finite unions and hence is a field.*

⟦Let E, $F \in \mathcal{M}$. Take any set C as a test set for E and take $C \cap E^c$ as a test set for F. This gives

$$\mu^*(C) = \mu^*(C \cap E) + \mu^*(C \cap E^c) \quad \text{and}$$
$$\mu^*(C \cap E^c) = \mu^*(C \cap E^c \cap F) + \mu^*(C \cap E^c \cap F^c).$$

Combining these we have

$$\begin{aligned} \mu^*(C) &= \mu^*(C \cap E) + \mu^*(C \cap E^c \cap F) + \mu^*(C \cap E^c \cap F^c) \\ &\geq \mu^*\big[(C \cap E) \cup (C \cap E^c \cap F)\big] + \mu^*(C \cap E^c \cap F^c) \qquad \text{(by subadditivity)} \\ &= \mu^*\big[C \cap (E \cup F)\big] + \mu^*\big[C \cap (E \cup F)^c\big]. \end{aligned}$$

Therefore, $E \cup F \in \mathcal{M}$.⟧

(2) *$C \subseteq X$, E, $F \in \mathcal{M}$ and $E \cap F = \emptyset \Rightarrow \mu^*\big(C \cap (E \cup F)\big) = \mu^*(C \cap E) + \mu^*(C \cap F)$.*

⟦Using $C \cap (E \cup F)$ as a test set for E we have

$$\begin{aligned} \mu^*\big[C \cap (E \cup F)\big] &= \mu^*\big[C \cap (E \cup F) \cap E\big] + \mu^*\big[C \cap (E \cup F) \cap E^c\big] \\ &= \mu^*(C \cap E) + \mu^*(C \cap F). \rrbracket \end{aligned}$$

(3) *If the sets E_n are disjoint, then $F := \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$ and $\mu(F) = \sum_{n=1}^{\infty} \mu(E_n)$.*

⟦Let $F_n := \bigcup_{k=1}^{n} E_k$ and $C \subseteq X$. By steps (1) and (2) and induction, $F_n \in \mathcal{M}$ and $\mu^*(C \cap F_n) = \sum_{k=1}^{n} \mu^*(C \cap E_k)$. Therefore, by monotonicity,

$$\mu^*(C) = \mu^*(C \cap F_n) + \mu^*(C \cap F_n^c) \geq \sum_{k=1}^{n} \mu^*(C \cap E_k) + \mu^*(C \cap F^c)$$

for all n and so

$$\mu^*(C) \geq \sum_{k=1}^{\infty} \mu^*(C \cap E_k) + \mu^*(C \cap F^c) \geq \mu^*(C \cap F) + \mu^*(C \cap F^c) \geq \mu^*(C).$$

This shows that $F \in \mathcal{M}$. Taking $C = F$ verifies countable additivity.⟧

(4) *If $E_n \in \mathcal{M}$, then $\bigcup_n E_n \in \mathcal{M}$.*

⟦Use (1), (3) and (1.2).⟧ □

Exercises

1.53 Define an outer measure μ^* on $\mathcal{P}(X)$ by $\mu^*(\emptyset) = 0$ and $\mu^*(E) = 1$ if $E \neq \emptyset$. Find $\mathcal{M}(\mu^*)$.

1.54 Let $\mathcal{O}_I$ denote the collection of all bounded open subintervals of $\mathbb{R}$ and let $\mu := \delta_0$ be the Dirac measure at 0 on $\mathcal{O}_I$. Show that the outer measure μ^* generated by $(\mathcal{O}_I, \mu)$ is the Dirac measure at 0 on $\mathcal{P}(\mathbb{R})$. Find $\mathcal{M}(\mu^*)$.

1.55 Let X be an uncountable set and define $\mu^*(E) = 0$ if $E = \emptyset$ and $\mu^*(E) = 1$ otherwise. Show that $\mu^*(E) = 0$ or 1 according as E is countable or uncountable. Show also that $\mathcal{M}(\mu^*)$ is the σ-field of sets that are countable or cocountable.

1.56 [$\uparrow$1.3.3(f)] Let μ be a monotone set function on a field $\mathcal{F}$. For $E \in \mathcal{F}$, let μ_E denote the set function on $\mathcal{F}$ defined by $\mu_E(A) = \mu(E \cap A)$ and let $(\mu_E)^*$ be the outer measure generated by $(\mathcal{F}, \mu_E)$. Prove that $(\mu^*)_E = (\mu_E)^*$.

1.57 [$\downarrow$1.8.1.] Let $\mathcal{A}$ and $\mathcal{B}$ be pavings of X such that each contains sequence with union X. Let μ be a measure on $\mathcal{A} \cup \mathcal{B}$ and let μ_a^* and μ_b^* be the outer measures generated by $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \mu)$, respectively. Suppose that

$$\mu_a^*(E) = \mu_b^*(E) = \mu(E) \ \forall\ E \in \mathcal{A} \cup \mathcal{B}. \tag{$\dagger$}$$

Prove that $\mu_a^* = \mu_b^*$. Show that assertion fails if the condition in ($\dagger$) is not assumed.

1.58 Let μ^* be an outer measure on X, $E \subseteq X$, and $A \in \mathcal{M}(\mu^*)$ with $E \cap A = \emptyset$. Show that $\mu^*(E \cup A) = \mu^*(E) + \mu(A)$.

1.59 Let μ^* be an outer measure on X, $E \subseteq X$, and A, $B \in \mathcal{M}(\mu^*)$ with $A \cap B = \emptyset$. Show that $\mu^*\big(E \cap (A \cup B)\big) = \mu^*(E \cap A) + \mu(E \cap B)$. Show that the conclusion holds for countable disjoint unions as well.

1.60 Let μ a nonnegative set function on a paving $\mathcal{A}$ of X with $\mu(\emptyset) = 0$, and let μ^* be the outer measure generated by $(\mathcal{A}, \mu)$. Prove that $E \in \mathcal{M}(\mu^*)$ for any $E \subseteq X$ satisfying

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \ \text{ for all } A \in \mathcal{A}.$$

1.6 Extension of a Measure

We have seen that a suitably defined pair $(\mathcal{A}, \mu)$ generates an outer measure μ^* and that the restriction of μ^* to the σ-field $\mathcal{M}(\mu^*)$ of measurable sets is a complete measure. A more intimate connection between μ and μ^* is possible if certain additional conditions are imposed on $(\mathcal{A}, \mu)$. For this we need the following definitions.

A nonempty collection $\mathcal{A}$ of subsets X is called a

- **semiring** if $\mathcal{A}$ is a π-system and for any A, $B \in \mathcal{A}$, there exist finitely many disjoint members C_j of $\mathcal{A}$ with $A \setminus B = \bigcup_{j=1}^n C_j$.
- **ring** if and A, $B \in \mathcal{A}$ implies $A \cup B$, $A \setminus B \in \mathcal{A}$.

Every ring is a π system and hence a semiring, since $A \cap B = A \setminus (A \setminus B)$ The collection of all bounded intervals on $\mathbb{R}$ is a semiring that is not a ring. A ring that contains X is closed under complements and hence is a field. If $(X, \mathcal{F}, \mu)$ is a measure space, then the collection of all members of $\mathcal{F}$ with finite measure is a ring that obviously need not be a field.

In this section we show that a measure μ on a semiring $\mathcal{A}$ may be extended to a measure on $\sigma(\mathcal{A})$ and that under suitable conditions the extension is unique and possesses certain approximation and completeness properties.

The Measure Extension Theorem

Let $\mathcal{A}$ be a semiring on a set X, μ a measure on $\mathcal{A}$, and μ^* the outer measure generated by $(\mathcal{A}, \mu)$. The proof of the measure extension theorem is based on the following lemmas.

1.6.1 Lemma. *The set $\mathcal{A}_\mathfrak{u}$ of all finite disjoint unions of members of $\mathcal{A}$ is a ring.*

Proof. Let A, $B \in \mathcal{A}_\mathfrak{u}$, say

$$A = \bigcup_{j=1}^{m} A_j,\ A_j \in \mathcal{A}, \quad \text{and} \quad B = \bigcup_{k=1}^{n} B_k,\ B_k \in \mathcal{A} \quad \text{(disjoint unions).}$$

To see that $A \setminus B \in \mathcal{A}_\mathfrak{u}$, for each j and k choose finitely many disjoint sets $C_{ijk} \in \mathcal{A}$ such that $A_j \setminus B_k = \bigcup_i C_{ijk}$. Then $A_j \setminus B_k \in \mathcal{A}_\mathfrak{u}$ and

$$A \setminus B = \bigcup_{j=1}^{m} A_j \cap B^c = \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} A_j \setminus B_k = \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} \bigcup_{i} C_{ijk}.$$

Since this is a disjoint union of members of $\mathcal{A}$, $A \setminus B \in \mathcal{A}_\mathfrak{u}$.

To show that $A \cup B \in \mathcal{A}_\mathfrak{u}$, write $A \cup B$ as the disjoint union $(A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ and note that $A \cap B$ is the disjoint union $\bigcup_{j,\,k} A_j \cap B_k$ of members of $\mathcal{A}$. □

1.6.2 Lemma. *Define a set function $\mu_\mathfrak{u}$ on $\mathcal{A}_\mathfrak{u}$ by*

$$\mu_\mathfrak{u}\left(\bigcup_{j=1}^{m} A_j\right) = \sum_{j=1}^{m} \mu(A_j), \quad A_j \in \mathcal{A} \quad \text{(disjoint union).}$$

Then $\mu_\mathfrak{u}$ is a well-defined measure on $\mathcal{A}_\mathfrak{u}$ and $\mu_\mathfrak{u}\big|_{\mathcal{A}} = \mu$.

Proof. To show that $\mu_\mathfrak{u}$ is well-defined, let $\bigcup_{j=1}^{m} A_j = \bigcup_{k=1}^{n} B_k$ be disjoint unions of members of $\mathcal{A}$. Then $A_j = \bigcup_{k=1}^{n} A_j \cap B_k$ and $B_k = \bigcup_{j=1}^{m} A_j \cap B_k$, hence

$$\mu(A_j) = \sum_{k=1}^{n} \mu(A_j \cap B_k) \quad \text{and} \quad \mu(B_k) = \sum_{j=1}^{m} \mu(A_j \cap B_k).$$

Summing, we obtain

$$\sum_{j=1}^{m} \mu(A_j) = \sum_{j=1}^{m} \sum_{k=1}^{n} \mu(A_j \cap B_k) = \sum_{k=1}^{n} \mu(B_k).$$

To show countable additivity, let $E_1, E_2 \ldots \in \mathcal{A}_\mathfrak{u}$ be disjoint with union $E \in \mathcal{A}_\mathfrak{u}$. Choose disjoint sets $A_1, \ldots, A_m \in \mathcal{A}$ such that $E = \bigcup_{i=1}^{m} A_i$, and for each k choose disjoint sets $B_{k,1}, \ldots, \in B_{k,m_k} \in \mathcal{A}$ such that $E_k = \bigcup_{j=1}^{m_k} B_{k,j}$. Then

$$E_k = E \cap E_k = \bigcup_{j=1}^{m} A_i \cap E_k = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m_k} A_i \cap B_{k,j}, \quad \text{(disjoint unions).}$$

By definition of $\mu_\mathfrak{u}$,

$$\mu_\mathfrak{u}(E) = \sum_{i=1}^{m} \mu(A_i) \quad \text{and} \quad \mu_\mathfrak{u}(E_k) = \sum_{i=1}^{m} \sum_{j=1}^{m_k} \mu(A_i \cap B_{k,j}). \tag{α}$$

Also, for each i,

$$A_i = A_i \cap E = \bigcup_{k=1}^{\infty} A_i \cap E_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} A_i \cap B_{k,j} \quad \text{(disjoint unions)},$$

hence, by the countable additivity of μ,

$$\mu(A_i) = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \mu(A_i \cap B_{k,j}). \tag{β}$$

By (α) and (β) and a rearrangement,

$$\mu_{\mathfrak{u}}(E) = \sum_{i=1}^{m} \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \mu(A_i \cap B_{k,j}) = \sum_{k=1}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{m_k} \mu(A_i \cap B_{k,j}) = \sum_{k=1}^{\infty} \mu_{\mathfrak{u}}(E_k). \qquad \square$$

1.6.3 Lemma. *The outer measures generated by* $(\mathcal{A}, \mu)$ *and* $(\mathcal{A}_{\mathfrak{u}}, \mu_{\mathfrak{u}})$ *are the same.*

Proof. Let $E \subseteq X$. Typical sums in the definitions of $\mu(E)$ and $\mu_{\mathfrak{u}}(E)$ are, respectively,

$$s = \sum_{n=1}^{\infty} \mu(A_n),\ A_n \in \mathcal{A},\ E \subseteq \bigcup_{n=1}^{\infty} A_n, \text{ and } t = \sum_{n=1}^{\infty} \mu_{\mathfrak{u}}(B_n),\ B_n \in \mathcal{A}_{\mathfrak{u}},\ E \subseteq \bigcup_{n=1}^{\infty} B_n.$$

Since $\mathcal{A} \subseteq \mathcal{A}_{\mathfrak{u}}$, every sum s is also a sum t. On the other hand, since each B_n is a finite disjoint union of members of $\mathcal{A}$ and $\mu_{\mathfrak{u}}$ is additive, every t may be decomposed and written as an s. The infima over these sums are therefore the same. $\square$

We may now prove

1.6.4 Theorem. *Let* $\mathcal{A}$ *be a semiring on a set* X*,* μ *a measure on* $\mathcal{A}$*,* μ^* *the outer measure generated by* $(\mathcal{A}, \mu)$*, and* $\mathcal{M} = \mathcal{M}(\mu^*)$ *the* σ*-field of* μ^**-measurable sets. Then* $\sigma(\mathcal{A}) \subseteq \mathcal{M}$ *and the measure* $\mu^*\big|_{\mathcal{M}}$ *is an extension of* μ*.*[1]

Proof. By the last lemma, we may assume that $\mathcal{A}$ is a ring. To show that $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$, let $A \in \mathcal{A}$ and $C \subseteq X$. We show that

$$\mu^*(C \cap A) + \mu^*(C \cap A^c) \le \mu^*(C). \tag{$\dagger$}$$

Let $C_n \in \mathcal{A}$ such that $C \subseteq \bigcup_{n=1}^{\infty} C_n$. Since $\mathcal{A}$ is a ring, $C_n \cap A$, $C_n \cap A^c \in \mathcal{A}$. Moreover, $C \cap A \subseteq \bigcup_{n=1}^{\infty}(C_n \cap A)$ and $C \cap A^c \subseteq \bigcup_{n=1}^{\infty}(C_n \cap A^c)$, so

$$\mu^*(C \cap A) \le \sum_{n=1}^{\infty} \mu(C_n \cap A) \quad \text{and} \quad \mu^*(C \cap A^c) \le \sum_{n=1}^{\infty} \mu(C_n \cap A^c).$$

Adding we have

$$\mu^*(C \cap A) + \mu^*(C \cap A^c) \le \sum_{n=1}^{\infty} \mu(C_n \cap A) + \sum_{n=1}^{\infty} \mu(C_n \cap A^c) = \sum_{n=1}^{\infty} \mu(C_n).$$

Since the cover (C_n) of C was arbitrary, ($\dagger$) holds.

To show that $\mu^*\big|_{\mathcal{A}} = \mu$, let A, $A_n \in \mathcal{A}$ with $A \subseteq \bigcup_{n=1}^{\infty} A_n$. Then

$$\mu(A) \le \sum_{n=1}^{\infty} \mu(A \cap A_n) \le \sum_{n=1}^{\infty} \mu(A_n).$$

Taking infima over all such sequences $\{A_n\}$ yields $\mu(A) \le \mu^*(A)$. On the other hand, the sequence $A, \emptyset, \emptyset, \ldots$ is a cover of A by members of $\mathcal{A}$, hence $\mu^*(A) \le \mu(A)$. Therefore, $\mu^*\big|_{\mathcal{A}} = \mu$, completing the proof of the theorem. $\square$

[1] We frequently denote this extension also by μ, depending on context.

Approximation Property of the Extension

1.6.5 Theorem. *Let $E \in \sigma(\mathcal{A})$ with $\mu(E) < \infty$. Then for each $\varepsilon > 0$ there exist disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ such that*

$$\mu\Big(E \triangle \bigcup\nolimits_{j=1}^{n} A_j\Big) < \varepsilon.$$

Proof. Choose a cover $\{B_n\}$ of E in $\mathcal{A}$ such that $\sum_n \mu(B_n) < \mu(E) + \varepsilon/2$. Define

$$E_1 = B_1 \quad\text{and}\quad E_n = B_n \cap B_1^c \cdots \cap B_{n-1}^c = (B_n \setminus B_1) \cap \cdots \cap (B_n \setminus B_{n-1}), \quad n \geq 2.$$

The sets E_n are disjoint and cover E. Choose n so large that $\sum_{j=n+1}^{\infty} \mu(B_j) < \varepsilon/2$. From the inclusion

$$E \triangle \bigcup_{j=1}^{n} E_j = \left[\left(\bigcup_{j=1}^{n} E_j\right) \setminus E\right] \cup \left[E \setminus \bigcup_{j=1}^{n} E_j\right] \subseteq \left[E^c \cap \bigcup_{j=1}^{\infty} E_j\right] \cup \bigcup_{j=n+1}^{\infty} E_j$$

we have

$$\begin{aligned}\mu\left(E \triangle \bigcup_{j=1}^{n} E_j\right) &\leq \mu\left(E^c \cap \bigcup_{j=1}^{\infty} E_j\right) + \mu\left(\bigcup_{j=n+1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(B_j) - \mu(E) + \sum_{j=n+1}^{\infty} \mu(E_j) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Noting that each E_j is a disjoint union of members of $\mathcal{A}$ (because $\mathcal{A}$ is a semiring), we obtain the desired approximation. □

Completeness of the Extension

1.6.6 Theorem. *If $(\mathcal{A}, \mu)$ is σ-finite, then $\mathcal{M}(\mu^*)$ is the completion of $\big(\sigma(\mathcal{A}), \mu\big)$.*

Proof. Let $\mathcal{F} = \sigma(\mathcal{A})$. Since $\mathcal{M}(\mu^*)$ is complete, by minimality $\mathcal{F}_\mu \subseteq \mathcal{M}(\mu^*)$. For the reverse inclusion, assume first that $\mu(X) < \infty$. Let $E \in \mathcal{M}(\mu^*)$ and for each n choose sequences $\{A_{n,j}\}_{j=1}^{\infty}$ and $\{B_{n,j}\}_{j=1}^{\infty}$ in $\mathcal{A}$ such that

$$\mu^*(E^c) \leq \mu(A_n) \leq \sum_{j=1}^{\infty} \mu(A_{n,j}) \leq \mu^*(E^c) + 1/n, \quad \text{where} \quad A_n := \bigcup_{j=1}^{\infty} A_{n,j} \supseteq E^c, \quad \text{and}$$

$$\mu^*(E) \leq \mu(B_n) \leq \sum_{j=1}^{\infty} \mu(B_{n,j}) \leq \mu^*(E) + 1/n, \quad \text{where} \quad B_n := \bigcup_{j=1}^{\infty} B_{n,j} \supseteq E.$$

Then B_n, $A_n^c \in \sigma(\mathcal{A})$, $A_n^c \subseteq E \subseteq B_n$, and

$$\mu(B_n) \to \mu^*(E), \quad \mu(A_n^c) = \mu(X) - \mu(A_n) \to \mu(X) - \mu^*(E^c) = \mu^*(E). \tag{†}$$

Next, let

$$A = \bigcup_{n=1}^{\infty} A_n^c \quad\text{and}\quad B = \bigcap_{n=1}^{\infty} B_n.$$

Then $A_n^c \subseteq A \subseteq E \subseteq B \subseteq B_n$, hence from (†), $\mu(B \setminus A) = 0$. Setting $M = B \setminus A$ and $N = E \setminus A$ we have $E = A \cup N$, $N \subseteq M$, A, $M \in \mathcal{F}$, and $\mu(M) = 0$ and so $E \in \mathcal{F}_\mu$.

In the general case, let $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$. Set $\mathcal{A}_n = \mathcal{A} \cap X_n$ and $\mu_n = \mu\big|_{\mathcal{A}_n}$. Then $\mathcal{A}_n$ is a semiring on X_n (Ex. 1.62) and μ_n is a measure on $\mathcal{A}_n$, so the outer measure μ_n^* generated by $(\mathcal{A}_n, \mu_n)$ is a measure on $\mathcal{F}_n := \sigma(\mathcal{A}_n) = \mathcal{F} \cap X_n$ with completion $\mathcal{M}(\mu_n^*)$. By Ex. 1.62 again, $\mathcal{M}(\mu_n^*) = \mathcal{M}(\mu^*) \cap X_n$ and μ_n^* is the restriction of μ^* to $\mathcal{M}(\mu_n^*)$. Now let $E \in \mathcal{M}(\mu^*)$. By the preceding paragraph, for each n there exist M_n, $A_n \in \mathcal{F}_n$ with $\mu_n(M_n) = 0$ and $N_n \subseteq M_n$ such that $E \cap X_n = A_n \cup N_n$. Setting

$$A = \bigcup_{n=1}^{\infty} A_n, \quad M = \bigcup_{n=1}^{\infty} M_n, \quad \text{and} \quad N = \bigcup_{n=1}^{\infty} N_n,$$

we have $E = A \cup N$, $N \subseteq M$, M, $A \in \mathcal{F}$ and $\mu(M) = 0$, hence $E \in \mathcal{F}_\mu$. □

1.6.7 Remark. The σ-finite hypothesis in the completeness theorem cannot be removed. For example, let $\mathcal{A} = \{\emptyset, \mathbb{R}\}$ with $\mu(\emptyset) = 0$ and $\mu(\mathbb{R}) = \infty$. Then $\mu^*(C) = \infty$ for any $C \neq \emptyset$, hence, trivially, $\mu(C) = \mu(C \cap E) + \mu(C \cap E^c)$ for all $E \subseteq \mathbb{R}$, that is, $\mathcal{M}(\mu^*) = \mathcal{P}(\mathbb{R})$. On the other hand, since the only set of measure zero is the empty set, the completion of $\sigma(\mathcal{A}) = \mathcal{A}$ is just $\mathcal{A}$. ◊

Uniqueness of the Extension

Uniqueness is an immediate consequence of the following more general result:

1.6.8 Theorem. *Let $(Y, \mathcal{P})$ be a π-system and let μ_1 and μ_2 be measures on $\sigma(\mathcal{P})$ that are σ-finite on $\mathcal{P}$. If $\mu_1\big|_{\mathcal{P}} = \mu_2\big|_{\mathcal{P}}$, then $\mu_1 = \mu_2$.*

Proof. The proof uses Dynkin's π-λ theorem. Suppose first that $Y \in \mathcal{P}$ and $\mu_1(Y) < \infty$. Let $\mathcal{L} = \{E \in \sigma(\mathcal{P}) : \mu_2(E) = \mu_1(E)\}$. We claim that $\mathcal{L}$ is a λ-system. Indeed, property (a) of (1.5) holds by assumption and (c) holds by continuity from below. To verify (b), let A, $B \in \mathcal{L}$ with $A \subseteq B$. Then

$$\mu_2(B \setminus A) = \mu_2(B) - \mu_2(A) = \mu_1(B) - \mu_1(A) = \mu_1(B \setminus A),$$

verifying the claim. Since $\mathcal{P} \subseteq \mathcal{L}$, by Dynkin's theorem $\sigma(\mathcal{P}) \subseteq \mathcal{L}$. This proves the theorem for the case $\mu_1(Y) < \infty$.

Now let (Y_n) be a disjoint sequence in $\mathcal{P}$ with union Y and $\mu_1(Y_n) < \infty$ for all n. Applying the result of the first paragraph to the restriction of the measures to Y_n, we see that $\mu_1(A \cap Y_n) = \mu_2(A \cap Y_n)$ for all $A \in \sigma(\mathcal{P})$ and all n. Now use countable additivity to complete the proof. □

Applying Theorem 1.6.8 to the current setting we have

1.6.9 Theorem. *If $(\mathcal{A}, \mu)$ is σ-finite, then the extension of μ to $\sigma(\mathcal{A})$ is unique.*

1.6.10 Remarks. Without the σ-finite hypothesis the conclusion of 1.6.9 may fail. For example, let $\mathcal{A}$ be the semiring of all bounded intervals and take μ to be the measure on $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{A})$ that assigns the value ∞ to every nonempty set in $\mathcal{A}$ (hence $\mu^*(E) = \infty$ for every nonempty $E \subseteq \mathbb{R}$). If ν is counting measure on $\mathcal{B}(\mathbb{R})$, then $\mu \neq \nu$, yet the measures agree on $\mathcal{A}$. Note also that μ (vacuously) has the approximation property, but ν does not.

The conclusion of 1.6.9 may also fail if $\mathcal{A}$ is not a semiring. For example, let $\mathcal{A}$ be the collection of all intervals $(a, b]$ with $b - a = 1$. If $\mu(A)$ is the number of integers in $A \in \mathcal{B}(\mathbb{R})$ and λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$ (see §1.7), then $\mu = \lambda$ on $\mathcal{A}$ but not on $\mathcal{B}(\mathbb{R})$. ◊

The following consequence of 1.6.8 will be needed later.

1.6.11 Theorem. *Let ν be any measure on $\sigma(\mathcal{A})$ that is σ-finite on $\mathcal{A}$. Then*

$$\nu(E) = \inf\left\{\sum_{n=1}^{\infty} \nu(A_n) : \ A_n \in \mathcal{A} \text{ and } E \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, \quad E \in \sigma(\mathcal{A}).$$

Proof. Let ν^* denote the outer measure generated by $(\nu|_{\mathcal{A}}, \mathcal{A})$. Then the measures $\nu^*|_{\sigma(\mathcal{A})}$ and ν agree on the π-system $\mathcal{A}$ and so are equal. □

Exercises

1.61 Let $\mathcal{A}_i$ be a semiring on X_i, $i = 1, 2$. Show that $\mathcal{A}_1 \times \mathcal{A}_2$ is a semiring.

1.62 Let μ be a measure on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ and $E \in \mathcal{A}$

(a) Prove that $\mathcal{A} \cap E$ is a semiring consisting of the members of $\mathcal{A}$ that are subsets of E.

(b) Let ν be the restriction of μ to $\mathcal{A} \cap E$ and let μ^* and ν^* be the outer measures generated by $(X, \mathcal{A}, \mu)$ and $(E, \mathcal{A} \cap E, \nu)$. Show that ν^* is the restriction of μ^* to $\mathcal{P}(E)$.

(c) Prove that $\mathcal{M}(\nu^*) = \mathcal{M}(\mu^*) \cap E$.

1.63 Let μ be as in 1.6.4 and let ν be a measure on $\sigma(\mathcal{A})$ that equals μ on $\mathcal{A}$.

(a) Show that $\nu(E) \leq \mu(E)$ for all $E \in \sigma(\mathcal{A})$. (1.6.10 shows equality may not hold.)

(b) Show that $\nu(E) = \mu(E)$ for all $E \in \sigma(\mathcal{A})$ with $\mu(E) < \infty$. ⟦ Assume that $\mathcal{A}$ is a ring (how?). Choose $A \in \mathcal{A}$ such that $E \subseteq A$ and $\mu(A) < \mu(E) + \varepsilon$. Then $\nu(E) + \nu(A \setminus E) < \nu(E) + \varepsilon$. ⟧

1.64 Let μ be a measure on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ and let μ^* be the outer measure generated by $(\mathcal{A}, \mu)$. Prove that for any $E \subseteq X$ there exists $A \in \sigma(\mathcal{A})$ such that $E \subseteq A$ and $\mu^*(E) = \mu(A)$.

1.65 [$\uparrow$1.64] Let μ be a measure on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ and let μ^* be the outer measure generated by $(\mathcal{A}, \mu)$. Prove the *weak inclusion-exclusion principle*

$$\mu^*(E \cup F) + \mu^*(E \cap F) \leq \mu^*(E) + \mu^*(F), \quad E,\, F \subseteq X.$$

1.66 [$\uparrow$1.64] Let μ and ν be measures on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ and let μ^* and ν^* be the outer measures generated by $(\mathcal{A}, \mu)$ and $(\mathcal{A}, \nu)$, respectively. Prove that $(\mu + \nu)^* = \mu^* + \nu^*$ and $\mathcal{M}(\mu^*) \cap \mathcal{M}(\nu^*) \subseteq \mathcal{M}(\mu^* + \nu^*)$. Show that the inclusion may be strict.

1.67 [$\uparrow$1.64, 1.40] Let μ and μ_n be σ-finite measures on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ with $\mu_n \uparrow \mu$ on $\sigma(\mathcal{A})$. Let μ^*, μ_n^* be the outer measures generated by $(\mathcal{A}, \mu)$ and $(\mathcal{A}, \mu_n)$ Prove that $\mu_n^* \uparrow \mu^*$ on $\mathcal{P}(X)$.

1.68 [$\uparrow$1.66, 1.67] Let μ_n be measures on a semiring $\mathcal{A}$ on X and define $\mu(A) = \sum_{n=1}^\infty \mu_n(A)$ $(A \in \mathcal{A})$. Let μ^* and μ_n^* be the outer measures generated by $(\mathcal{A}, \mu)$ and $(\mathcal{A}, \mu_n)$, respectively. Prove that $\mu^* = \sum_{n=1}^\infty \mu_n^*$.

1.69 [$\uparrow$1.64] Let μ be a measure on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ and let μ^* be the outer measure generated by $(\mathcal{A}, \mu)$. Prove that μ^* is continuous from below. Why doesn't this imply that μ^* is a measure on $\mathcal{P}(X)$?

1.70 [$\uparrow$1.64] Let μ be a measure on a semiring $\mathcal{A} \subseteq \mathcal{P}(X)$ and let μ^* be the outer measure generated by $(\mathcal{A}, \mu)$. Suppose that $\mu^*(X) < \infty$. Show that $E \in \mathcal{M}(\mu^*)$ iff $\mu(X) = \mu^*(E) + \mu^*(E^c)$.

1.7 Lebesgue Measure

The Volume Set Function

Recall that $\mathcal{H}_I$ denotes the semiring of bounded, left open d-dimensional intervals

$$(\boldsymbol{a}, \boldsymbol{b}] := (a_1, b_1] \times \cdots \times (a_d, b_d], \quad \boldsymbol{a} := (a_1, \ldots, a_d), \quad \boldsymbol{b} := (b_1, \ldots, b_d),$$

where $-\infty < a_j \leq b_j < \infty$. Define the **$d$-dimensional volume** of $(\boldsymbol{a}, \boldsymbol{b}]$ by

$$\lambda(\boldsymbol{a}, \boldsymbol{b}] = \lambda^d(\boldsymbol{a}, \boldsymbol{b}] := \prod_{j=1}^d (b_j - a_j).$$

In this section we apply the results of §1.6 to the pair $(\mathcal{H}_I, \lambda)$ to construct **d-dimensional Lebesgue measure**. The following lemma is key to the construction.

1.7.1 Lemma. *Let $H, H_1, \ldots, H_m \in \mathcal{H}_I$.*

(a) *If $H_1, \ldots, H_m$ are disjoint and $H = \bigcup_{j=1}^m H_j$, then $\lambda(H) = \sum_{j=1}^m \lambda(H_j)$.*

(b) *If $H \subseteq \bigcup_{j=1}^m H_j$, then $\lambda(H) \le \sum_{j=1}^m \lambda(H_j)$.*

(c) *If $H_1, \ldots, H_m$ are disjoint and $H \supseteq \bigcup_{j=1}^m H_j$, then $\lambda(H) \ge \sum_{j=1}^m \lambda(H_j)$.*

Proof. For ease of notation and exposition, we prove the lemma for $d = 2$, in which case the intervals are rectangles. Let $H = (a, b] \times (c, d]$. We may assume in (b) that $H = \bigcup_{j=1}^m H_j$,

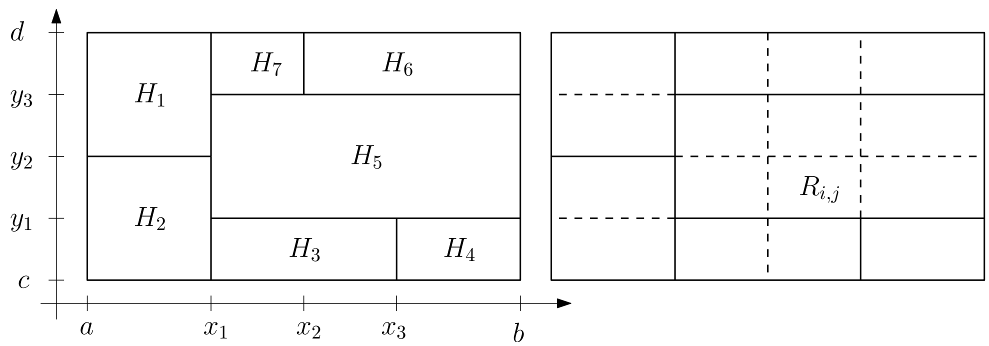

FIGURE 1.1: Pairwise disjoint interval grid of H.

otherwise replace H_j by $H_j \cap H$. Thus in each case the rectangles H_j are contained in H, hence the coordinates of their vertices form partitions

$$\{x_0 := a < x_1 < \ldots < x_p := b\} \quad \text{and} \quad \{y_0 := c < y_1 < \ldots < y_q := d\}$$

of $[a, b]$ and $[c, d]$, respectively. These partitions generate a grid of disjoint subrectangles $R_{i,j} = (x_i, x_{i+1}] \times (y_j, y_{j+1}]$ with union H such that each H_k is a union of such subrectangles. The procedure for case (a) is illustrated in Figure 1.1. Since

$$b - a = \sum_{i=0}^{p-1}(x_{i+1} - x_i) \quad \text{and} \quad d - c = \sum_{j=1}^{q-1}(y_{j+1} - y_j),$$

we have, by the definition of λ,

$$\lambda(H) = \left[\sum_{i=0}^{p-1}(x_{i+1} - x_i)\right]\left[\sum_{j=0}^{q-1}(y_{j+1} - y_j)\right] = \sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\lambda(R_{i,j}). \tag{1.9}$$

Similarly,

$$\lambda(H_k) = \sum_{(i,j):R_{i,j} \subseteq H_k} \lambda(R_{i,j})$$

so that

$$\sum_k \lambda(H_k) = \sum_k \sum_{(i,j):R_{i,j} \subseteq H_k} \lambda(R_{i,j}). \tag{1.10}$$

Now compare (1.9) and (1.10). In (a), every $R_{i,j}$ is contained in exactly one H_k, hence the rectangles in (1.10) appear exactly once and so $\lambda(H) = \sum_{k=1}^m \lambda(H_k)$. In (b), a rectangle $R_{i,j}$ could be contained in more than one H_k, so $\lambda(H) \le \sum_{k=1}^m \lambda(H_k)$. Finally, in (c) not every $R_{i,j}$ is necessarily contained in an H_k, hence $\lambda(H) \ge \sum_{k=1}^m \lambda(H_k)$. $\square$

1.7.2 Lemma. *The volume set function λ is countably additivity on $\mathcal{H}_I$.*

Proof. Part (a) of 1.7.1 gives finite additivity. Let $\{H_j\}$ be a sequence of disjoint members of $\mathcal{H}_I$ such that $H := \bigcup_{j=1}^{\infty} H_j \in \mathcal{H}_I$. By 1.7.1(c), $\lambda(H) \geq \sum_{j=1}^{n} \lambda(H_j)$ for all n, hence $\lambda(H) \geq \sum_{j=1}^{\infty} \lambda(H_j)$.

For the reverse inequality, let $\varepsilon > 0$, and for each j let H_j^{ε} denote the member of $\mathcal{H}_I$ obtained by replacing each coordinate subinterval $(c, d]$ of H_j by $(c - \delta_j, d + \delta_j]$, where δ_j is chosen so that $\lambda(H_j^{\varepsilon}) < \lambda(H_j) + \varepsilon/2^j$. Then the collection of intervals $\operatorname{int} H_j^{\varepsilon}$ is an open covering of the compact set $\operatorname{cl} H$, so there exists an $m \in \mathbb{N}$ such that $H \subseteq \operatorname{int} H_1^{\varepsilon} \cup \cdots \cup \operatorname{int} H_m^{\varepsilon} \subseteq H_1^{\varepsilon} \cup \cdots \cup H_m^{\varepsilon}$. By 1.7.1(b),

$$\lambda(H) - \varepsilon < \lambda(H^{\varepsilon}) \leq \lambda(H_1^{\varepsilon}) + \cdots + \lambda(H_m^{\varepsilon}) \leq \sum_{j=1}^{\infty} \lambda(H_j) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\lambda(H) \leq \sum_{j=1}^{\infty} \lambda(H_j)$, establishing countable additivity. □

Construction of the Measure

Since $\sigma(\mathcal{H}_I) = \mathcal{B}(\mathbb{R}^d)$, we may invoke 1.6.4 using the outer measure

$$\lambda^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \lambda(A_n) : \ A_n \in \mathcal{H}_I \text{ and } E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}, \quad E \subseteq \mathbb{R}^d,$$

to obtain

1.7.3 Theorem. *The volume set function λ on $\mathcal{H}_I$ has a unique extension to $\mathcal{B}(\mathbb{R}^d)$. Moreover, $\mathcal{M}(\lambda^*)$ is the completion of $\mathcal{B}(\mathbb{R}^d)$.*

The members of $\mathcal{M}(\lambda^*)$ are called **Lebesgue measurable sets** and $\lambda := \lambda^*\big|_{\mathcal{M}(\lambda^*)}$ is called **Lebesgue measure** on $\mathbb{R}^d$.

Exercises

1.71 Let $I \in \mathcal{H}_I$. Show that $\lambda(I) = \lambda(\operatorname{int} I) = \lambda(\operatorname{cl} I)$. Also, in the definition

$$\lambda^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \lambda(A_n) : \ A_n \in \mathcal{A} \text{ and } E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}, \quad E \subseteq \mathbb{R}^d,$$

where $\mathcal{A} = \mathcal{H}_I$, show that the infimum is unchanged if $\mathcal{A}$ is taken to be $\mathcal{O}_I$, $\mathcal{C}_I$, $\mathcal{O} :=$ the set of open sets of $\mathbb{R}^d$, or $\mathcal{K} :=$ the set of compact subsets of $\mathbb{R}^d$.

1.72 Let $N \subseteq \mathbb{R}^d$ with $\lambda(N) = 0$. Show that N^c is dense in $\mathbb{R}$.

1.73 (*Translation invariance of* λ). Let $E \subseteq \mathbb{R}^d$ and $\boldsymbol{x} \in \mathbb{R}^d$. Show that

(a) $\lambda^*(\boldsymbol{x} + E) = \lambda^*(E)$ (b) $E \in \mathcal{M}(\lambda) \Rightarrow \boldsymbol{x} + E \in \mathcal{M}(\lambda)$.

1.74 (*Dilation property of* λ). Let $E \subseteq \mathbb{R}^d$ and $r \in \mathbb{R}$. Show that

(a) $\lambda^*(rE) = |r|^d \lambda^*(E)$ (b) $E \in \mathcal{M}(\lambda) \Rightarrow rE \in \mathcal{M}(\lambda)$.

1.75 Show that for any $\varepsilon > 0$ there exists an open set U dense in $\mathbb{R}^d$ such that $\lambda(U) < \varepsilon$.

1.76 Let A, $B \subseteq [0, 1]$, where $B \in \mathcal{M}(\lambda)$ and $\lambda(B) = 1$. Show that $\lambda^*(A) = \lambda^*(A \cap B)$.

1.77 Let $E \subseteq \mathbb{R}$ with $0 < \lambda(E) < \infty$ and let $0 < r < 1$. Show that there exists an interval $[a, b]$ such that $\lambda^*(E \cap [a, b]) > r(b - a)$. ⟦Let I_n be closed, bounded intervals that cover E with $\sum_n \lambda(I_n) < r^{-1} \lambda(E)$. ⟧

1.78 Show that the graph $G := \{(x, f(x)) : x \in \mathbb{R}\}$ of a continuous function f is a Borel set with two-dimensional Lebesgue measure zero.

1.8 Lebesgue-Stieltjes Measures

A measure on $\mathcal{B}(\mathbb{R}^d)$ that is finite on bounded, d-dimensional intervals is called a **Lebesgue-Stieltjes measure**. For example, Lebesgue measure λ^d is a Lebesgue-Stieltjes measure. Lebesgue-Stieltjes measures may be constructed from so-called *distribution functions*, discussed below. Before we describe the construction, we discuss some approximation properties possessed by these measures.

Regularity

The following theorem complements the approximation property 1.6.5.

1.8.1 Theorem. *Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}^d$ and let $E \in \mathcal{B}(\mathbb{R}^d)$. Then*

(a) $\mu(E) = \inf\{\mu(U) : U$ *open and* $U \supseteq E\}$.

(b) $\mu(E) = \sup\{\mu(K) : K$ *compact and* $K \subseteq E\}$.

Proof. Assume first that E is bounded. Let $\varepsilon > 0$. By 1.6.11 (taking $\mathcal{A} = \mathcal{O}_I$, say), there exists a sequence of bounded, open, d-dimensional intervals I_j with union $U \supseteq E$ such that $\mu(U) \le \sum_j \mu(I_j) < \mu(E) + \varepsilon$, verifying (a).

To verify (b) in the bounded case, let J be a bounded open interval with $\mathrm{cl}(E) \subseteq J$. Choose a sequence of open intervals V_k with union $V \supseteq J \setminus E$ such that $\sum_{k=1}^{\infty} \mu(V_k) < \mu(J \setminus E) + \varepsilon/2$. We may assume that $V_k \subseteq J$, otherwise replace V_k by $V_k \cap J$. By subadditivity

$$\mu(V) \le \sum_{k=1}^{\infty} \mu(V_k) \le \mu(J \setminus E) + \varepsilon/2 = \mu(J) - \mu(E) + \varepsilon/2.$$

Set $K = J \setminus V$. Since $K \subseteq E \subseteq \mathrm{cl}(E) \subseteq J$, $K = \mathrm{cl}(E) \setminus V$. Therefore, K is compact and $\mu(K) = \mu(J) - \mu(V) \ge \mu(J) - \big(\mu(J) - \mu(E) + \varepsilon/2\big) = \mu(E) - \varepsilon/2$, verifying (b).

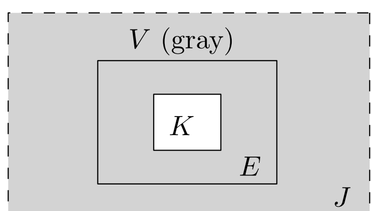

FIGURE 1.2: Construction of K.

Now suppose E is unbounded. Choose a sequence of bounded sets $E_n \in \mathcal{M}(\mu)$ such that $E_n \uparrow E$. Let $\varepsilon > 0$. For each n, use the first part of the proof to choose a compact set K_n and an open set U_n with finite measure such that

$$K_n \subseteq E_n \subseteq U_n, \quad \mu(U_n) - \mu(E_n) < \varepsilon/2^n \quad \text{and} \quad \mu(E_n) - \mu(K_n) < \varepsilon.$$

Set $U := \bigcup_{n=1}^{\infty} U_n$. Then U is open, $E \subseteq U$, and $U \setminus E \subseteq \bigcup_n (U_n \setminus E_n)$. If $\mu(E) < \infty$, then

$$\mu(U) - \mu(E) = \mu(U \setminus E) \le \sum_n \mu(U_n \setminus E_n) < \varepsilon,$$

and for sufficiently large n,

$$\mu(E) - \mu(K_n) = \mu(E \setminus E_n) + \mu(E_n \setminus K_n) < \varepsilon,$$

verifying (a) and (b) in this case. On the other hand, if $\mu(E) = \infty$, then (a) clearly holds and (b) holds as well because then $\mu(E_n) \uparrow \infty$ and $\mu(K_n) > \mu(E_n) - \varepsilon$. $\square$

One-Dimensional Distribution Functions

A nondecreasing, right continuous function $F : \mathbb{R} \to \mathbb{R}$ is called a **distribution function**. Such functions arise naturally in probability theory (see Chapter 18). The connection between Lebesgue-Stieltjes measures and distribution functions is described in the following theorem, the proof of which is given below.

1.8.2 Theorem. *For every Lebesgue-Stieltjes measure μ on $\mathbb{R}$, there exists a distribution function F such that*

$$\mu(a, b] = F(b) - F(a) \quad \textit{for all } a < b. \tag{1.11}$$

Any two distributions that satisfy (1.11) for the same μ differ by a constant. Conversely, every distribution function F gives rise to a unique Lebesgue-Stieltjes measure μ on $\mathcal{B}(\mathbb{R})$ satisfying (1.11).

Here are three common examples:

1.8.3 Examples.

(a) The Dirac measure δ_0 on $\mathcal{B}(\mathbb{R})$ has distribution function $F = \mathbf{1}_{[0,\infty)}$.

(b) Let (c_n) and (p_n) be sequences in $\mathbb{R}$ with $p_n > 0$ and $\sum_n p_n < \infty$. Define

$$F(x) = \sum_{n : c_n \le x} p_n,$$

where the sum is taken over all indices n for which $c_n \le x$. (If there are no such indices, the sum is defined to be 0.) Note that because the order of summation is irrelevant, F is well-defined. The Lebesgue-Stieltjes measure corresponding to F is given by

$$\mu(B) = \sum_{n : c_n \in B} p_n \quad \text{for all Borel sets } B.$$

The distribution in (a) is a special case, obtained by taking $p_1 = 1$, $p_n = 0$ for $n \ge 2$, and $c_n = 0$ for all n.

(c) Let f be continuous and nonnegative on $\mathbb{R}$. Define

$$F(x) = F(0) + \int_0^x f(t)\, dt,$$

where $F(0)$ is arbitrary. The Lebesgue-Stieltjes measure corresponding to F is $d\mu = f\, dt$. (See Chapter 3.) ♢

Proof of Theorem 1.8.2. For the first part of the theorem, define $F : \mathbb{R} \to \mathbb{R}$ as follows: Let $F(0)$ be arbitrary and set

$$F(x) := \begin{cases} F(0) + \mu(0, x] & \text{if } x > 0, \\ F(0) - \mu(x, 0] & \text{if } x < 0. \end{cases}$$

By considering cases, we see that for $a < b$, $F(b) - F(a) = \mu(a, b]$. Therefore, F is nondecreasing and right continuous. If also $G(b) - G(a) = \mu(a, b]$ for all $a < b$, then $F(x) - F(0) = G(x) - G(0)$ for all x, hence $F = G + F(0) - G(0)$.

For the converse, let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. To construct the Lebesgue-Stieltjes measure defined by F, we apply the results of §1.6 to $(\mathcal{H}_I, \mu)$, where μ is the set function on $\mathcal{H}_I$ given by (1.11). Thus the proof of the theorem will be complete if we show that μ is countably additive on $\mathcal{H}_I$. The following lemmas, analogous to those of §1.7, establish this.

1.8.4 Lemma. *Let $H, H_1, \ldots, H_m \in \mathcal{H}_I$.*

(a) *If $H_1, \ldots, H_m$ are disjoint and $H = \bigcup_{j=1}^m H_j$, then $\mu(H) = \sum_{j=1}^m \mu(H_j)$.*

(b) *If $H \subseteq \bigcup_{j=1}^m H_j$, then $\mu(H) \le \sum_{j=1}^m \mu(H_j)$.*

(c) *If $H_1, \ldots, H_m$ are disjoint and $H \supseteq \bigcup_{j=1}^m H_j$, then $\mu(H) \ge \sum_{j=1}^m \mu(H_j)$.*

Proof. Let $H = (a, b]$ and $H_j = (a_j, b_j]$, where $a_1 < a_2 < \cdots < a_m$. In (a) there can be no "gaps" or "overlaps," that is, $a_1 = a$, $b_m = b$, and $b_j = a_{j+1}$. Therefore,

$$\sum_{j=1}^m \mu(H_j) = \sum_{j=1}^{m-1} [F(a_{j+1}) - F(a_j)] + F(b) - F(a_m) = F(b) - F(a) = \mu(H).$$

In (b), we may assume that $H = \bigcup_{j=1}^m H_j$, otherwise we could replace H_j by $H_j \cap H$. As in (a), $a_1 = a$, $b_m = b$, and $a_{j+1} \le b_j$. However, since the intervals are no longer disjoint it may happen that $a_{j+1} < b_j$ for some j, as illustrated in Figure 1.3. Form intersections of overlapping intervals, thus partitioning $(a, b]$ into a collection $\{I_i\}$ of disjoint half-open intervals, as shown in the figure. Each H_j is a union of some of these intervals so by (a)

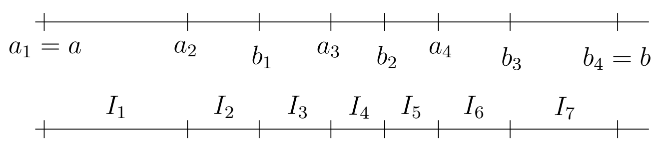

FIGURE 1.3: Construction of partition.

$$\mu(H) = \sum_i \mu(I_i) \quad \text{and} \quad \mu(H_j) = \sum_{i : I_i \subseteq H_j} \mu(I_i).$$

Since an I_i may be contained in more than one H_j

$$\sum_i \mu(I_i) \le \sum_j \sum_{i : I_i \subseteq H_j} \mu(I_i).$$

Therefore, $\mu(H) \le \sum_j \mu(H_j)$, proving (b). The proof of (c) is similar. $\square$

1.8.5 Lemma. *The set function μ is countably additive on $\mathcal{H}_I$.*

Proof. By 1.8.4(a), μ is finitely additive. Let $H_j = (a_j, b_j]$ be disjoint members of $\mathcal{H}_I$ and let $H = (a, b] = \bigcup_{j=1}^\infty H_j$. By 1.8.4(c), $\mu(H) \ge \sum_{j=1}^m \mu(H_j)$ for all m, hence $\mu(H) \ge \sum_{j=1}^\infty \mu(H_j)$. For the reverse inequality, let $\varepsilon > 0$ and by right continuity at a choose $r \in (a, b)$ so that $F(r) \le F(a) + \varepsilon/2$. Then

$$\mu(r, b] = F(b) - F(r) \ge F(b) - F(a) - \varepsilon/2 = \mu(H) - \varepsilon/2. \tag{†}$$

Similarly, for each j choose $r_j > b_j$ such that $F(r_j) \le F(b_j) + \varepsilon/2^j$, so

$$\mu(a_j, r_j] = F(r_j) - F(a_j) \le F(b_j) - F(a_j) + \varepsilon/2^j = \mu(H_j) + \varepsilon/2^j. \tag{‡}$$

The open intervals (a_j, r_j) cover $[r, b]$, hence by compactness there exists an $m \in \mathbb{N}$ such that $(r, b] \subseteq \bigcup_{j=1}^m (a_j, r_j]$. By (†), (‡), and 1.8.4(b),

$$\mu(H) \le \varepsilon/2 + \mu(r, b] \le \varepsilon/2 + \sum_{j=1}^\infty \mu(a_j, r_j] \le \varepsilon + \sum_{j=1}^\infty \mu(H_j).$$

Letting $\varepsilon \to 0$ yields the desired inequality. $\square$

***Higher Dimensional Distribution Functions**

As in the one-dimensional case, there is a close connection between Lebesgue-Stieltjes measures on $\mathcal{B}(\mathbb{R}^d)$ and certain real-valued functions on $\mathbb{R}^d$. The technical details are more intricate, however, and depend on the following construct:

The ***i*th coordinate difference operator** on functions $F : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\triangle_{a_i}^{b_i} F(x_1, \ldots, x_d) = F(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_d) - F(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_d).$$

For example, consider the function $F(x_1, x_2, \ldots, x_d) = x_1 x_2 \ldots x_d$. For $1 \le i \le d$ and $a_i < b_i$, the difference operators may be applied successively to obtain the following:

$$\begin{aligned}
\triangle_{a_1}^{b_1} \cdots \triangle_{a_d}^{b_d} F(x_1, x_2, \ldots, x_d) &= \triangle_{a_1}^{b_1} \cdots \triangle_{a_{d-1}}^{b_{d-1}} (x_1 \cdots x_{d-1})(b_d - a_d) \\
&= \triangle_{a_1}^{b_1} \cdots \triangle_{a_{d-2}}^{b_{d-2}} (x_1 \cdots x_{d-2})(b_{d-1} - a_{d-1})(b_d - a_d) \\
&\ \ \vdots \\
&= (b_1 - a_1) \cdots (b_d - a_d).
\end{aligned}$$

Thus $\triangle_{a_1}^{b_1} \cdots \triangle_{a_d}^{b_d} F(x_1, x_2, \ldots, x_d)$ is the Lebesgue measure of the d-dimensional interval $(a_1, b_1] \times \cdots \times (a_d, b_d]$. This sort of connection holds more generally and is described in the theorem below. For the statement of the theorem we need the following definitions:

A function $F : \mathbb{R}^d \to \mathbb{R}$ is a **distribution function** if it is **nondecreasing** in the sense that

$$\triangle_{a_1}^{b_1} \cdots \triangle_{a_d}^{b_d} F(x_1, \ldots, x_d) \ge 0, \quad a_i < b_i, \ i = 1, \ldots, d,$$

and **right continuous** in the sense that

$$x_{i,n} \downarrow_n x_i, \quad i = 1, \ldots, d \ \Rightarrow F\big(x_{n,1}, \ldots, x_{n,d}\big) \to F(x_1, \ldots, x_d).$$

Here are some standard distribution functions:

1.8.6 Examples.

(a) Let F_i be a distribution function on $\mathbb{R}$, $i = 1, \ldots, d$. The function

$$F(x_1, x_2, \ldots, x_d) := F_1(x_1) F_2(x_2) \cdots F_d(x_d)$$

is a distribution function on $\mathbb{R}^d$ such that

$$\triangle_{a_1}^{b_1} \cdots \triangle_{a_d}^{b_d} F(x_1, \ldots, x_d) = \prod_{i=1}^{d} [F_i(b_i) - F_i(a_i)].$$

The function $F(x_1, x_2, \ldots, x_d) = x_1 x_2 \cdots x_d$ discussed above is a special case.

(b) Let f be a nonnegative, continuous function on $\mathbb{R}^d$. Then

$$F(x_1, \ldots, x_d) := \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(t_1, \ldots, t_d)\, dt_d \cdots dt_1$$

is a distribution function on $\mathbb{R}^d$ (provided the improper integral is finite) such that

$$\triangle_{a_1}^{b_1} \cdots \triangle_{a_d}^{b_d} F(x_1, \ldots, x_d) = \int_{a_1}^{b_1} \cdots \int_{a_d}^{b_d} f(t_1, \ldots, t_d)\, dt_d \cdots dt_1.$$

(c) If μ is a finite measure on $\mathcal{B}(\mathbb{R}^d)$, then

$$F(x_1, \ldots, x_d) = \mu((-\infty, x_1] \times \cdots \times (-\infty, x_d])$$

defines a distribution function on $\mathbb{R}^d$. $\Diamond$

The following theorem may be proved using a combination of ideas developed earlier in the construction of Lebesgue measure and Lebesgue-Stieltjes measures. For a proof, the reader is referred to [1] or [5].

1.8.7 Theorem. *Let μ be a Lebesgue-Stieltjes measure on $\mathcal{B}(\mathbb{R}^d)$. Then there exists a function $F : \mathbb{R}^d \to \mathbb{R}$ such that for all $a_i < b_i$*

$$\mu\big((a_1, b_1] \times \cdots \times (a_d, b_d]\big) = \triangle_{a_1}^{b_1} \cdots \triangle_{a_d}^{b_d} F(x_1, \ldots, x_d). \tag{1.12}$$

Conversely, given a distribution function $F : \mathbb{R}^d \to \mathbb{R}$, there exists a unique Lebesgue-Stieltjes measure on $\mathcal{B}(\mathbb{R}^d)$ such that (1.12) holds for all $a_i < b_i$ $(i = 1, \ldots, d)$.

Exercises

1.79 Describe the Lebesgue-Stieltjes measure for each of the following distribution functions.

(a) $F(x) = \lfloor x \rfloor$, the greatest integer function.

(b) $F(x) = x\mathbf{1}_{[0,1)} + \mathbf{1}_{[1,\infty]}$.

1.80 Show that the sum of finitely many distribution functions and the product of finitely many nonnegative distribution functions are distribution functions.

1.81 Verify that the function in 1.8.3(b) is a distribution function. Prove also that F is left continuous at a iff $a \neq c_n$ for every n.

1.82 For any monotone function $F : \mathbb{R} \to \mathbb{R}$ and $-\infty \leq a < b \leq \infty$, define

$$F(a+) := \lim_{x \to a^+} F(x) \quad \text{and} \quad F(b-) := \lim_{x \to b^-} F(x)$$

and set

$$F(-\infty) := F((-\infty)+) \quad \text{and} \quad F(\infty) := F(\infty-).$$

Let F be a distribution function and μ the associated Lebesgue-Stieltjes measure. Prove the following, when defined:

(a) $\mu(a, b) = F(b-) - F(a)$.

(b) $\mu[a, b) = F(b-) - F(a-)$.

(c) $\mu[a, b] = F(b) - F(a-)$.

Prove also that $\mu\{x\} = 0$ iff F is continuous at x.

1.83 Let μ be a finite Lebesgue-Stieltjes measure on $\mathcal{B}(\mathbb{R})$ such that $\mu\big(\{x\}\big) = 0$ for all x. Show that any distribution function F corresponding to μ is uniformly continuous on $\mathbb{R}$.

1.84 Show that a monotone function $f : \mathbb{R} \to \mathbb{R}$ has countably many discontinuities. Conclude that if μ is a Lebesgue-Stieltjes measure, then there exist at most countably many $x \in \mathbb{R}$ such that $\mu(\{x\}) > 0$. ⟦For each $t \in \mathbb{R}$, define $a_t = \lim_{x \to t^-} f(x)$ and $b_t = \lim_{x \to t^+} f(x)$. Then $a_t < b_t$ iff f is discontinuous at t.⟧

1.85 Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}$ with a continuous distribution function and let $A \in \mathcal{B}(\mathbb{R})$ with $\mu(A) > 0$. Prove that for each $b \in (0, \mu(A))$ there exists a Borel set $B \subseteq A$ such that $\mu(B) = b$. ⟦Use the intermediate value theorem on $G(x) = \mu\big(A \cap [-n, x]\big)$ for suitable n⟧.

*1.9 Some Special Sets

In this section we construct subsets of $\mathbb{R}$ that illustrate some of the finer points of Lebesgue and Borel measurability.

An Uncountable Set with Lebesgue Measure Zero

The **Cantor ternary set** C is constructed as follows: Remove from $I := [0,1] = I_{0,1}$ the "middle third" open interval $(1/3, 2/3)$, leaving closed intervals $I_{1,1}$ and $I_{1,2}$ with union C_1 and total length $2/3$. Next, remove from each of the intervals $I_{1,1}$ and $I_{1,2}$ the middle third open interval, leaving closed intervals $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, and $I_{2,4}$ with union C_2 and total length $4/9 = (2/3)^2$. By induction, one obtains a decreasing sequence of closed sets $C_k = \bigcup_{j=1}^{2^k} I_{k,j}$ such that $\lambda(C_k) = (2/3)^k$. (See Figure 1.4.) Then $C := \bigcap_k C_k$ is closed and $\lambda(C) = 0$.

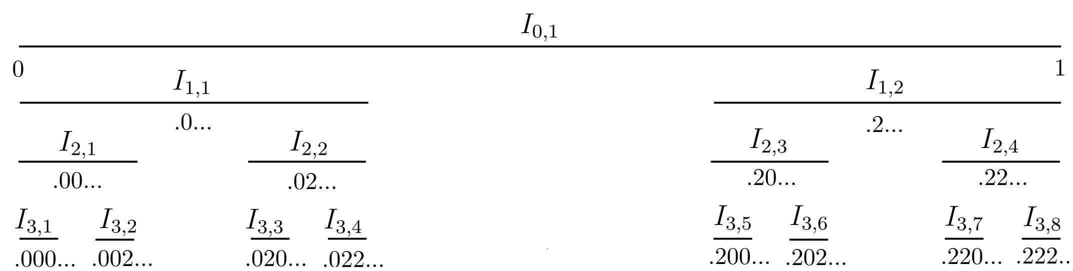

FIGURE 1.4: Middle thirds construction.

To show that C is uncountable, consider the ternary representation of a number $x \in [0,1]$:

$$x = .d_1 d_2 \ldots = \sum_{k=1}^{\infty} d_k 3^{-k}, \quad \text{where} \quad d_k \in \{0,1,2\}. \tag{1.13}$$

By induction, using the fact that $x \in I_{k-1,j} \Rightarrow I_{k,2j-1+d_k/2}$, one shows that $x \in C$ iff x has an expansion with even digits (see Figure 1.4). Define $\varphi : C \to [0,1]$ by

$$\varphi\big(.d_1 d_2 \ldots (\text{ternary})\big) = .e_1 e_2 \ldots (\text{binary}), \text{ where } d_k \in \{0,2\} \text{ and } e_k = d_k/2.$$

The function φ is not one-to-one, but by removing from C the countable set of all numbers with ternary representations ending in a sequence of zeros we obtain a set D on which φ is one-to-one. Since $\varphi(D) = (0,1)$, C is uncountable.

Non-Lebesgue-Measurable Sets

We show the following:

Every Lebesgue measurable set A with $\lambda(A) > 0$ contains a set that is not Lebesgue measurable.

Since $A = \bigcup_{n\in\mathbb{Z}} A \cap [n, n+1]$, we may suppose that A is bounded. Define an equivalence relation on A by $x \sim y$ iff $x - y \in \mathbb{Q}$. Let B be the subset of A obtained by choosing exactly one point from each distinct equivalence class. (The existence of B requires the axiom of choice.) Now observe that the sets $r + B$, $r \in \mathbb{Q}$, are disjoint. Indeed, if $(r+B) \cap (s+B) \neq \emptyset$, then $r + x = s + y$ for some x, $y \in B$, so $x = y$ and $r = s$. Moreover, since A is bounded so is $B + [0,1]$. Let (r_n) be an enumeration of the rationals in $[0,1]$ and assume that B is measurable. Then

$$\infty > \lambda\left(\bigcup_n (B + r_n)\right) = \sum_n \lambda(B + r_n) = \sum_n \lambda(B),$$

which implies that $\lambda(B) = 0$. But $A \subseteq B + \mathbb{Q}$, hence

$$\lambda(A) \leq \lambda\left(\bigcup_{r\in\mathbb{Q}} (B + r)\right) = \sum_{r\in\mathbb{Q}} \lambda(B + r) = 0,$$

contradicting that $\lambda(A) > 0$. Therefore, B cannot be Lebesgue measurable.

A Lebesgue Measurable, Non-Borel Set

For this example, we first construct the **Cantor function** $f : I \to I$, where $I = [0,1]$. The construction is based on the Cantor set C described earlier in the section. For each n, denote by $J_{n,k}$, $k = 1, \ldots, 2^{n-1}$, the open intervals in increasing order that were removed from I in the construction of C, that is, the intervals that form the complement of C_n in $[0,1]$. For example, $J_{2,1} = (1/9, 2/9)$, $J_{2,2} = (1/3, 2/3)$, and $J_{2,3} = (7/9, 8/9)$, hence

$$[0,1] = I_{2,1} \cup J_{2,1} \cup I_{2,2} \cup J_{2,2} \cup I_{2,3} \cup J_{2,3} \cup I_{2,4}.$$

Define a continuous function $f_n : I \to I$ so that

$$f_n(0) = 0, \quad f_n(1) = 1, \quad f_n = k/2^n \text{ on } J_{n,k},$$

and f_n is linear on the complementary intervals $I_{n,j}$. Since $|f_n(x) - f_{n+1}(x)| \leq 1/2^{n+1}$, the

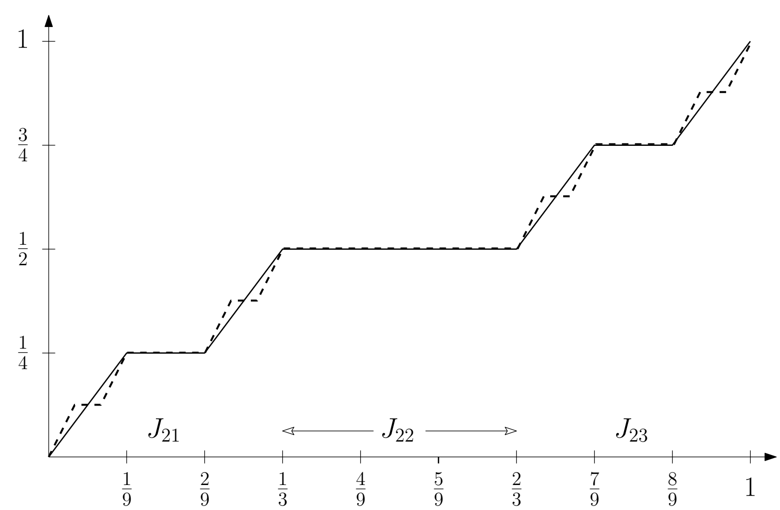

FIGURE 1.5: The functions f_2 and f_3.

sequence $\{f_n\}$ is uniformly Cauchy and so converges to a continuous function f, the Cantor function.

To construct the desired non-Borel set, note first that since $f_n(0) = 0$, $f_n(1) = 1$, and f_n is nondecreasing on $[0,1]$, f also has these properties. Thus, by the intermediate value theorem, $f(I) = I$. Since the values of f on the intervals $J_{n,k}$ are already assumed at the endpoints and since these endpoints lie in C, $f(J_{n,k})$ contributes nothing additional to the range of f, hence $f(C) = I$. Now set $g(x) = (f(x)+x)/2$, $x \in I$. Then g is continuous, *strictly* increasing, $g(0) = 0$, and $g(1) = 1$, hence $g(I) = I$. It follows that $g : I \to I$ is a homeomorphism, hence $g(C)$ is closed. Thus $g(I \setminus C)$ is a proper nonempty open subset of I and so has positive Lebesgue measure. Moreover, g takes the interval $J_{n,k}$, on which f is constant, to an open interval half its length, so by countable additivity $\lambda\big(g(I \setminus C)\big) = \lambda(I \setminus C)/2 = 1/2$ and therefore $\lambda(g(C)) = 1/2$. Now let E be a subset of $g(C)$ that is not Lebesgue measurable and let $A := g^{-1}(E)$. Then $A \subseteq C$ and so is Lebesgue measurable with $\lambda(A) = 0$. However, A cannot be a Borel set since g maps Borel sets onto Borel sets. (This is proved in Chapter 2.)

1.9.1 Remark. While the intricate nature of the construction of A might lead one to believe that such sets are rare, there are in fact many more Lebesgue measurable sets than Borel sets. Indeed, since the Cantor set C is uncountable and every subset of C is Lebesgue measurable, the collection of Lebesgue measurable sets has cardinality $2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of the continuum. On the other hand, it may be shown that $\mathcal{B}(\mathbb{R})$ has only cardinality $\mathfrak{c}$. (See, for example, [38].) $\Diamond$

Exercises

1.86 Show that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete.

1.87 Carry out the steps below to prove following assertion: If $A \subseteq \mathbb{R}$ has positive Lebesgue measure then the set $A - A := \{x - y : x,\, y \in A\}$ contains an interval $(-r, r)$ for some $r > 0$.

(a) Show that it suffices to consider the case A compact.

(b) Choose an open set $U \supseteq A$ such that $\lambda(U) < 2\lambda(A)$ (how?). Define a distance function $d : U \to \mathbb{R}$ by $d(x) = \inf\{|x - y| : y \in U^c\}$. Show that d is continuous and positive. Conclude that d has a minimum $r > 0$ on A.

(c) Show that $|x| < r \Rightarrow x + A \subseteq U \Rightarrow (x + A) \cap A \neq \emptyset$. Conclude that $(-r, r) \subseteq A - A$.

1.88 [$\uparrow$1.87] Show that the only subgroup of $(\mathbb{R}, +)$ that has positive Lebesgue measure is $\mathbb{R}$.

1.89 Let (a_n) be a sequence in $(0, 1)$ and set $b_n := 1 - a_n$. Mimic the construction of the Cantor ternary set by removing the middle part of $[0, 1]$ of length a_1, leaving two intervals with union E_1, each of length $b_1/2$, then removing the middle part of length $a_2 b_1/2$ from these leaving four intervals with union E_2, each of length $b_1 b_2/4$, and so forth. The intersection $E := \bigcap_n E_n$ is

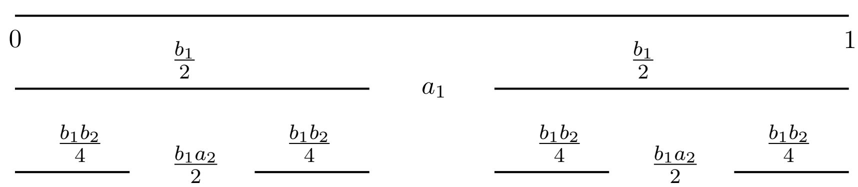

FIGURE 1.6: Generalized middle thirds construction.

called a **generalized Cantor set**. Verify the following:

(a) E is closed and $\lambda(E) = \displaystyle\prod_{n=1}^{\infty} b_n := \lim_n \prod_{j=1}^{n} b_j$.

(b) The interior of E is nonempty.

(c) If $r > 0$ and eventually $a_n \geq r$ (as in the Cantor ternary set), then $\lambda(E) = 0$.

(d) For each $a \in (0, 1)$, there exists a generalized Cantor set with Lebesgue measure a. [[Consider $\ln\big(\prod_{n=1}^{\infty} b_n\big) = \sum_{n=1}^{\infty} \ln b_n$.]]

1.90 Let A be the set of all $x \in [0, 1]$ having a decimal expansion $.d_1 d_2 \ldots$ with no digit equal to 3. Show that A is uncountable, $A \in \mathcal{B}(\mathbb{R})$, and $\lambda(A) = 0$.